ΤΕΧΝΟΛΟΓΙΚΟ ΕΚΠΑΙΔΕΥΤΙΚΟ ΙΔΡΥΜΑ ΚΡΗΤΗΣ
ΣΧΟΛΗ ΕΦΑΡΜΟΣΜΕΝΩΝ ΕΠΙΣΤΗΜΩΝ
ΤΜΗΜΑ ΗΛΕΚΤΡΟΝΙΚΗΣ

ΠΤΥΧΙΑΚΗ ΕΡΓΑΣΙΑ

ΔΟΜΗΣΗ ΗΛΕΚΤΡΟΝΙΚΩΝ ΔΙΑΛΕΞΕΩΝ ΜΕΤΑΠΤΥΧΙΑΚΟΥ ΜΑΘΗΜΑΤΟΣ ΣΤΗΝ ΨΗΦΙΑΚΗ ΕΠΕΞΕΡΓΑΣΙΑ ΣΗΜΑΤΟΣ

MSc LEVEL LECTURE DESIGN ON DIGITAL SIGNAL PROCESSING

Άννα Δασκαλάκη
(AM 4709)

Επιβλέπων Καθηγητής: Ιωάννης Π. Μακρής
Αναπλ. Καθηγητής

25/2/2015

Περίληψη

Σήμερα, οι μικροεπεξεργαστές και μικροελεγκτές έχουν φθάσει σε τόσο προχωρημένο στάδιο, ώστε να έχουν σημαντική επίπτωση στα επιστημονικά και τεχνολογικά πεδία των ηλεκτρονικών μηχανικών και των μηχανικών υπολογιστών. Συνεπώς, είναι σημαντικό οι μηχανικοί να εξοικειωθούν με τις έννοιες των ψηφιακών σημάτων και τις τεχνολογίες των ψηφιακών συστημάτων επικοινωνιών και ελέγχου και ακόμα πιο σημαντικό να εκμεταλλευτούν τις τεχνικές ψηφιακής επεξεργασίας σήματος (DSP).

Η παρούσα πτυχιακή εργασία σχεδιάζει ένα συμπαγές αλλά περιεκτικό σύνολο από διαφάνειες διαλέξεων για ένα, σε μεταπτυχιακό επίπεδο, εισαγωγικό μάθημα Ψηφιακής Επεξεργασίας Σήματος (DSP). Καλύπτει τις γενικές έννοιες του DSP, το θεώρημα δειγματοληψίας, τα αναλογικά anti-aliasing και anti-image χαμηλής διέλευσης φίλτρα, την ανασύσταση του αναλογικού σήματος, την μετατροπή από αναλογικό σε ψηφιακό, τα θεμελιώδη στοιχεία της αναλογικής επεξεργασίας σήματος (σειρά Fourier, μετασχηματισμοί Fourier και Laplace), τον διακριτό μετασχηματισμό Fourier (DFT) και τους αλγορίθμους του γρήγορου μετασχηματισμού Fourier και ολοκληρώνεται με μια επισκόπηση των φίλτρων Butterworth και Chebyshev. Όλη η έκταση του θεωρητικού περιεχομένου εμπλουτίζεται με ενδεικτικά παραδείγματα.

Abstract

Nowdays, microprocessors and microcontrollers have become so advanced that they have significant impact on the disciplines of electronics and computer engineering. Consequently, it is essential the engineers to become familiar with the concepts of digital signals and the technologies of digital systems for communications and controls and most important to exploit the digital signal processing (DSP) techniques.

The present graduate thesis designs compact yet comprehensive lecture slides for a postgraduate level introductory lesson on Digital Signal Processing (DSP). It covers the general concepts of DSP, the sampling theorem, analog anti-aliasing and anti-image lowpass filters, the signal reconstruction, analog-to-digital conversion, fundamentals of analog signal processing (Fourier series, Fourier and Laplace transforms), discrete Fourier transform (DFT) and fast Fourier transform (FFT) algorithms and ends with an overview of Butterworth and Chebyshev filters. The theoretical content is enriched with indicative examples.

Σύνοψη

Οι διαφάνειες 1 έως 5 εισάγουν τις βασικές έννοιες του DSP και παρουσιάζουν το σχηματικό διάγραμμα της ψηφιακής ανάλυσης σήματος. Επίσης, εξηγείται το ψηφιακό φιλτράρισμα και παρουσιάζεται η διαδικασία της φασματικής ανάλυσης σήματος.

Οι διαφάνειες 6 έως 13 καλύπτουν το θεώρημα δειγματοληψίας που περιγράφεται στο πεδίο του χρόνου και στο πεδίο της συχνότητας, παρουσιάζονται πρακτικές εκτιμήσεις για το σχεδιασμό αναλογικών anti-aliasing και anti-image χαμηλής διέλευσης φίλτρων και επεξηγείται η ανασύσταση του αναλογικού σήματος.

Οι διαφάνειες 14 έως 20 περιγράφουν τις αρχές μετατροπής σήματος αναλογικού σε ψηφιακό και ψηφιακού και σε αναλογικό και την κβαντοποίηση του σήματος. Στις διαφάνειες 21-23 ορίζονται τα γραμμικά, χρονικά αναλλοίωτα, αιτιατά συστήματα.

Στις διαφάνειες 24 έως 42 εκτελείται μία σύνοψη της αναλογικής επεξεργασίας σήματος, όπου παρουσιάζονται η περιγραφή περιοδικού σήματος με οι σειρές Fourier (sine-cosine form, amplitude-phase form, exponential form), οι μετασχηματισμοί Fourier και Laplace, η συνάρτηση μεταφοράς και η κρουστική απόκριση γραμμικού αναλογικού συστήματος, η ευστάθεια συστήματος, η συνέλιξη και η ημιτονική απόκριση σταθερής κατάστασης.

Στις διαφάνειες 43 έως 63 παρουσιάζονται συνοπτικά αναλογικά και ψηφιακά φίλτρα χαμηλής διέλευσης και ζώνης διέλευσης Butterworth και Chebyshev, με παραδείγματα.

Στις διαφάνειες 64 έως 79 καλύπτεται ο διακριτός μετασχηματισμός Fourier (DFT) και η ψηφιακή φασματική ανάλυση σήματος, ενώ εισάγεται η τεχνική του προσδιορισμού του φάσματος σήματος με τη χρήση παραθυρικών συναρτήσεων (window functions) με εφαρμογές.

Στις διαφάνειες 80 έως 88 περιγράφεται εποπτικά ο γρήγορος μετασχηματισμός Fourier (FFT) με τους αλγορίθμους του decimation-in-frequency και decimation-in-time.

Στις τελευταίες διαφάνειες παρατίθενται προβλήματα ανακεφαλαίωσης με τις λύσεις τους.

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

ΔΟΜΗΣΗ

ΗΛΕΚΤΡΟΝΙΚΩΝ ΔΙΑΛΕΞΕΩΝ

ΜΕΤΑΠΤΥΧΙΑΚΟΥ ΜΑΘΗΜΑΤΟΣ

ΣΤΗΝ

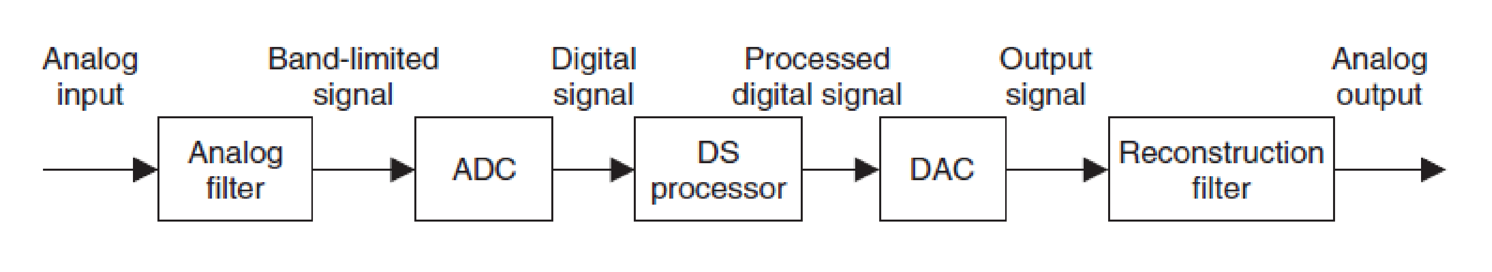
ΨΗΦΙΑΚΗ ΕΠΕΞΕΡΓΑΣΙΑ ΣΗΜΑΤΟΣ

MSC LEVEL LECTURE DESIGN

ON

DIGITAL SIGNAL PROCESSING

Anna Daskalaki | d_anna-13-ntora@windowslive.com



Digital Signal Processing (DSP) scheme

Digital Signal Processing (DSP) technology and its advancements have dramatically impacted our modern society everywhere.

- Internet (wire and wireless networks)
- digital audio and/or video
- digital recording/playing (CD, DVD, Blu Ray, MP3, MPEG4 etc)
- digital cameras
- digital and cellular telephones
- digital satellite and TV
- medical instruments

Without DSP, scientists, engineers, and technologists would have no powerful tools to analyze and visualize data and perform their design.

DSP Advantages

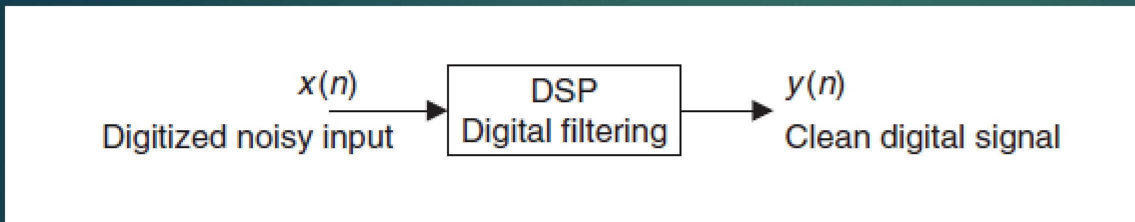
- ✓ minimum analog processing
- ✓ variety of processing algorithms
- ✓ flexibility
- ✓ less noise interference
- ✓ no signal distortion

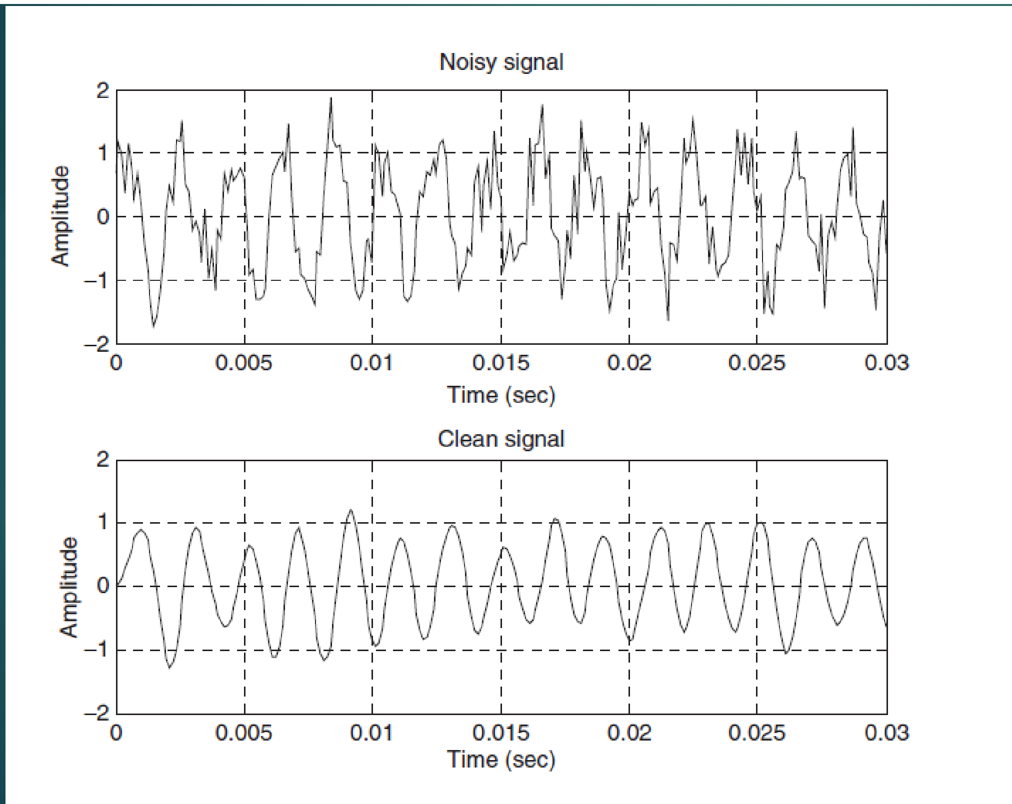
There are many real-world DSP applications that do not require DAC, such as data acquisition and digital information display, speech recognition, data encoding, etc.

Similarly, there are many real-world DSP applications that need no ADC, such as CD players, text-to-speech synthesis, digital tone generators etc.

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Digital Filtering

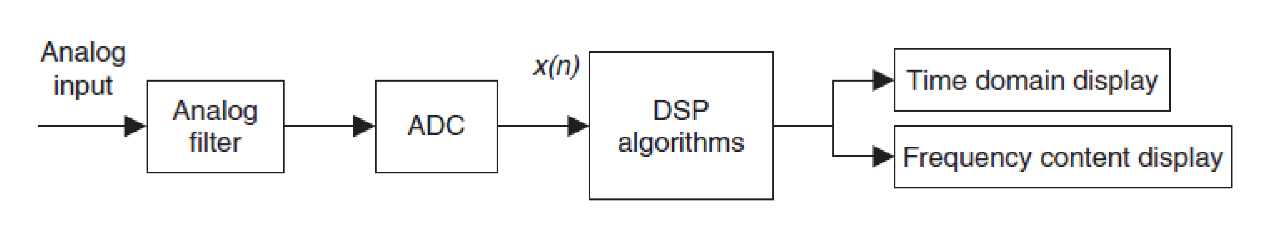


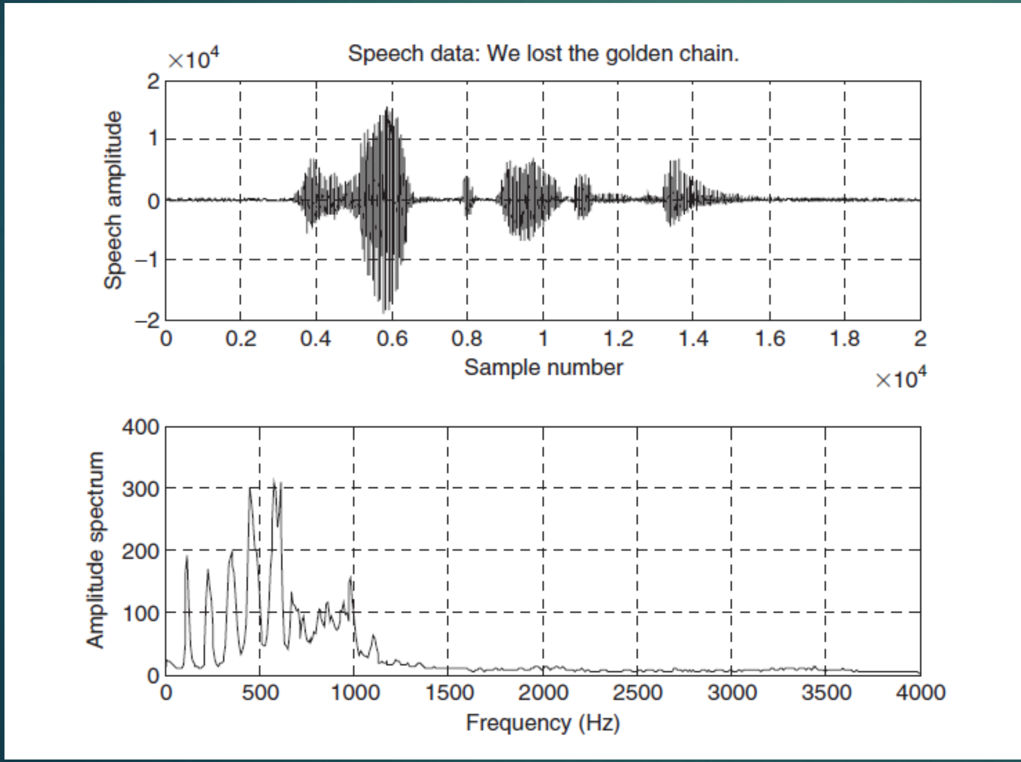


Since the useful signal contains the low-frequency component, the high frequency components above that of our useful signal are considered as noise, which can be removed by using a digital low-pass filter.

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Signal Frequency (Spectrum) Analysis





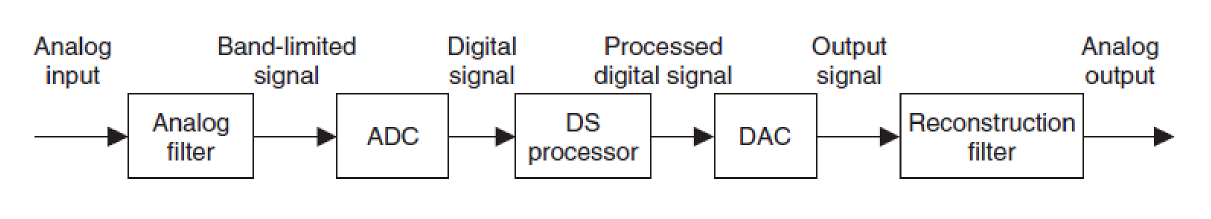
Spectral estimation of a digitally recorded speech waveform using the FFT algorithm

Digital speech waveform versus its digitized sample number, from a speech signal produced by a human in the time domain.

Frequency content information of speech for a range up to to 4kHz. It can be identified about ten speech formants, which can be used for applications such as speech modeling, speech coding, speech synthesis and recognition etc.

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Sampling of Continuous Signal

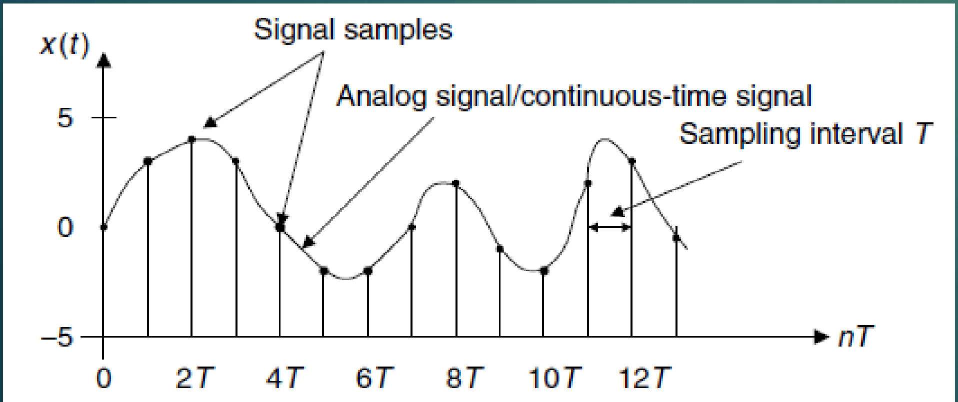


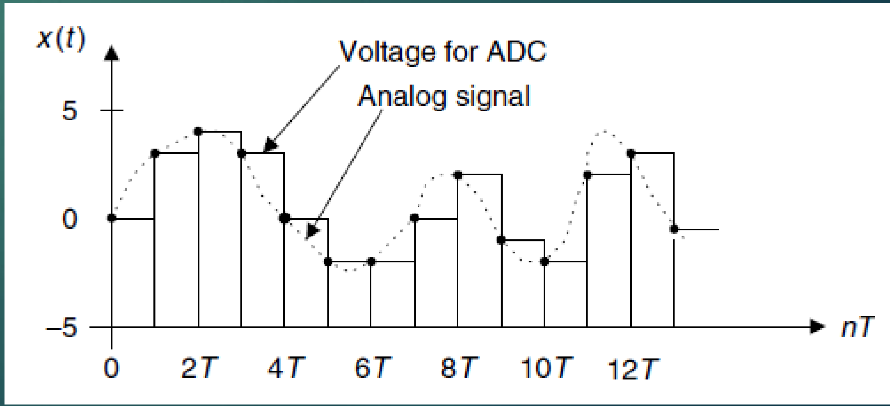
Digital Signal Processing System

The analog filter processes the analog input to obtain the band-limited signal, which is sent to the analog-to-digital conversion unit, which samples the analog signal, quantizes the sampled signal, and encodes the quantized signal levels to the digital signal.

Analog (continuous-time) signal, defined at every point over the time axis and amplitude axis, that is sampled at a fixed time interval, T, termed as the sampling period.

Sample-and-Hold analog voltage for ADC. Each sample maintains its voltage level during the sampling interval T to give the ADC enough time to convert it.



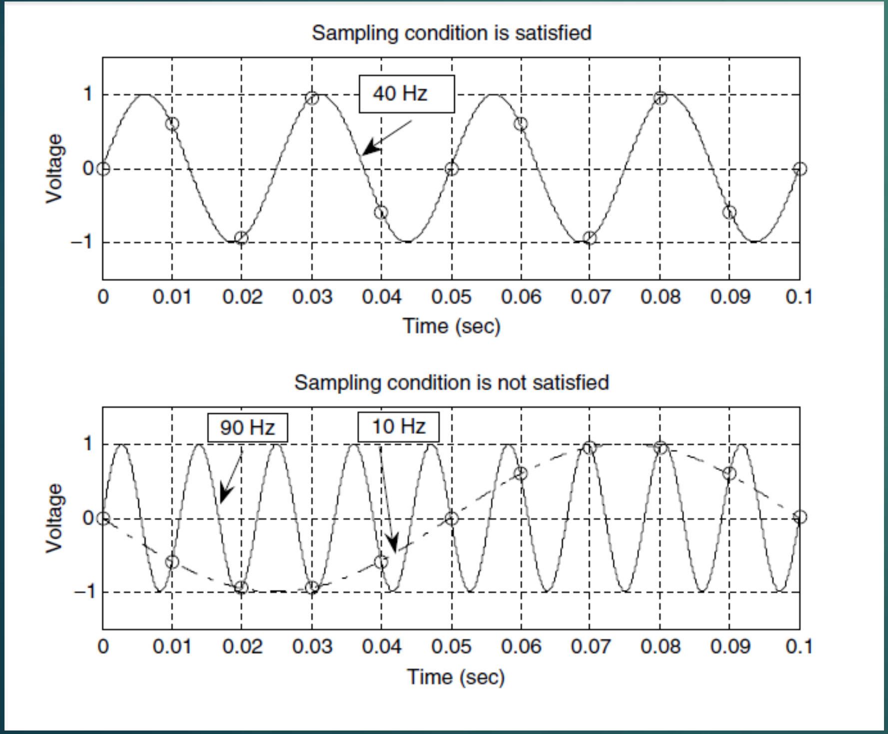


Sampling of Continuous Signal

For a given sampling interval T, which is defined as the time span between two sample points, the sampling rate (sampling frequency) is therefore given by:

$$f_s = \frac{1}{T} \text{ samples / second (Hz)}$$

appropriately sampled signals and non-appropriately sampled (aliased) signals.



Shannon sampling theorem: For a uniformly sampled DSP system, an analog signal can be perfectly recovered as long as the sampling rate is at least twice as large as the highest-frequency component of the analog signal to be sampled:

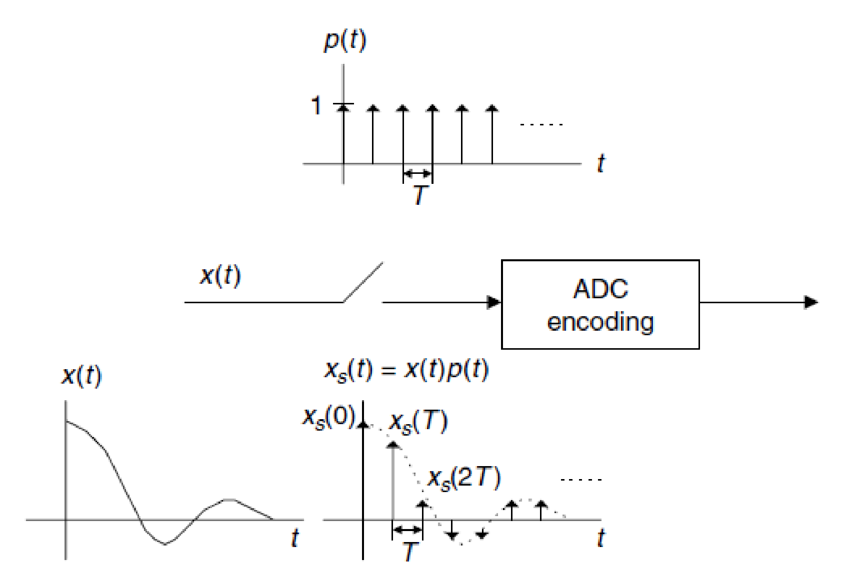
$$f_s \geq 2f_{max}$$

Half of the sampling frequency, $f_s/2$, is usually called the **Nyquist frequency** (Nyquist limit), or folding frequency.

The sampling theorem indicates that a DSP system with a sampling rate of fs can ideally sample an analog signal with its highest frequency up to half of the sampling rate without introducing spectral overlap (aliasing).

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Sampling of Continuous Signal
(frequency domain)



Sampled signal $x_s(t)$ obtained by sampling the continuous signal x(t) at a sampling rate of f_s samples per second.

Mathematically, this process can be written as the product of the continuous signal and the sampling pulses (pulse train):

$$x_{(s)}(t) = x(t)p(t)$$

with a period $T = 1/f_s$.

Sampled signal spectrum:

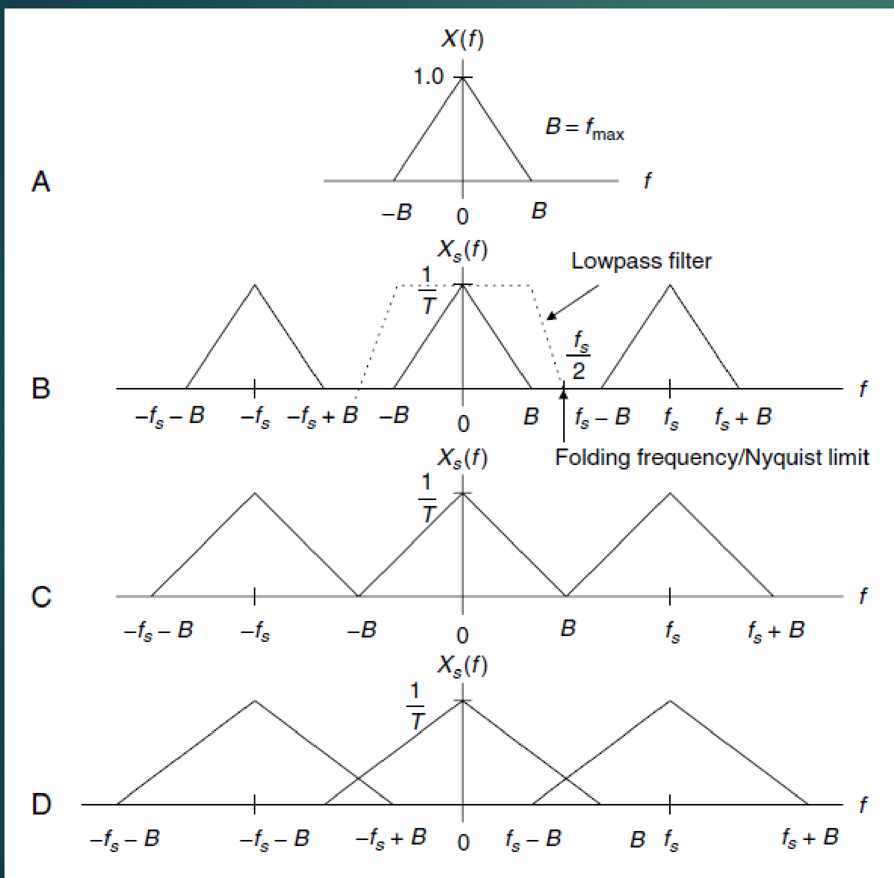
$$X_s(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(f - nf_s)$$

where *X(f)* the original baseband spectrum, while $X(f \pm nf_s)$ its replicas, so:

$$X_s(f) = \cdots + \frac{1}{T}X(f+f_s) + \frac{1}{T}X(f) + \frac{1}{T}X(f-f_s)\ldots$$

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Sampling of Continuous Signal
(frequency domain)



Given the original signal spectrum *X(f)* three possible sketches are classified for $X_s(f)$:

$\frac{1}{T}X(f+f_s), \frac{1}{T}X(f), \frac{1}{T}X(f-f_s)$ have separations between .

$\frac{1}{T}X(f+f_s), \frac{1}{T}X(f), \frac{1}{T}X(f-f_s)$ are just connected.

$\frac{1}{T}X(f+f_s), \frac{1}{T}X(f), \frac{1}{T}X(f-f_s)$ are overlapped .

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Example

uppose that an analog signal is given as

$x(t)=5\cos(2\pi\ 1000t)$ for $t \geq 0$

And is sampled at the rate of 8,000Hz

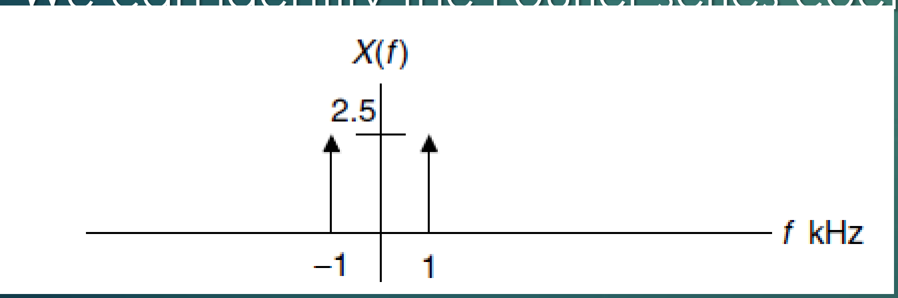
a. Sketch the spectrum for the original signal

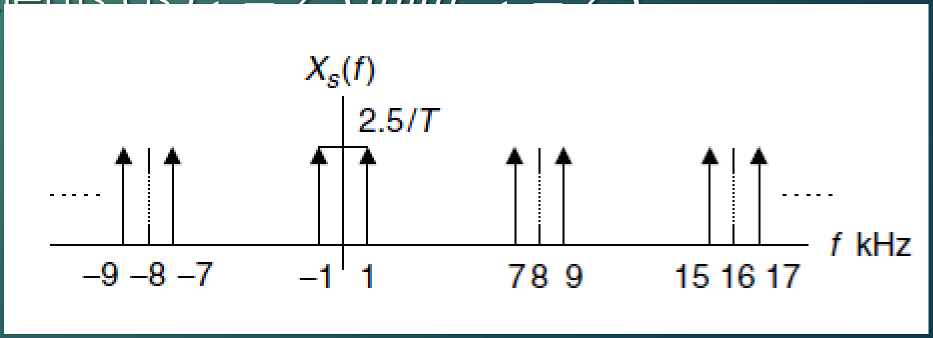
b. sketch the spectrum for the sampled signal from 0 to 20 kHz.

Solution: Since the analog signal is sinusoid with a peak value of 5 and frequency of 1,000Hz ,we can write the sine wave using Euler's identity:

$$5\cos(2\pi \times 1{,}000t) = 5\left(\frac{e^{j2\pi\times1000t}+e^{-j2\pi\times1000t}}{2}\right) = 2{,}5\ e^{j2\pi\times1000t} + 2{,}5\ e^{-j2\pi\times1000t}$$

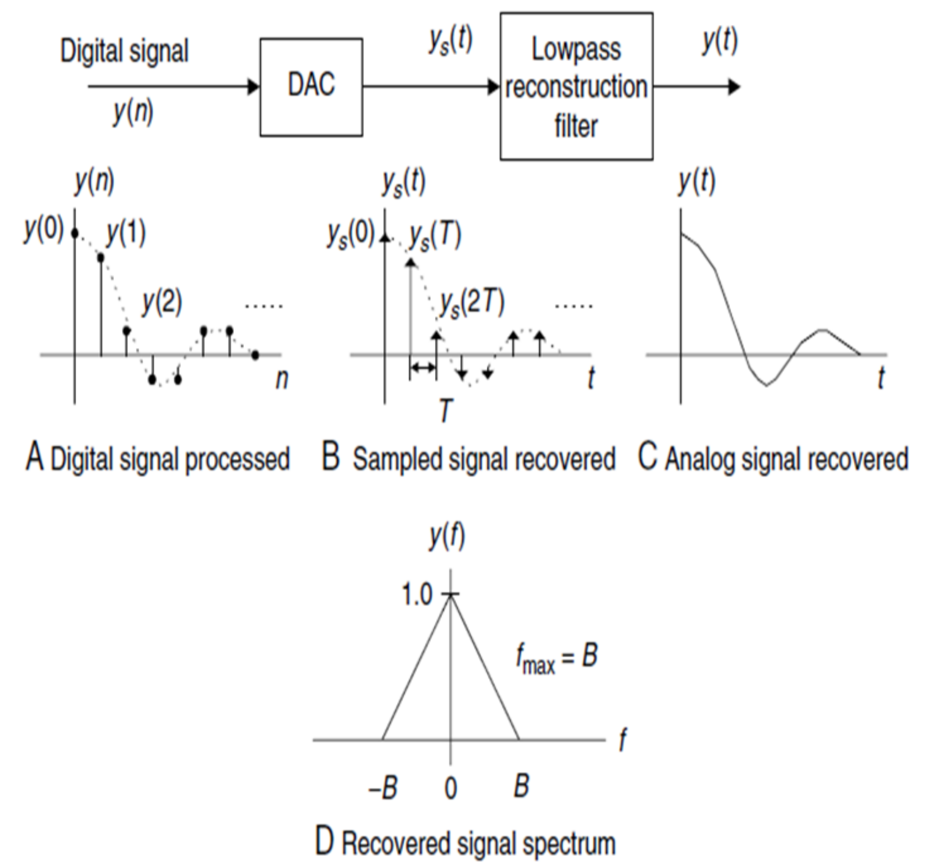
We can identify the Fourier series coefficients as $c_1 = 2.5$ and $c_{-1} = 2.5$





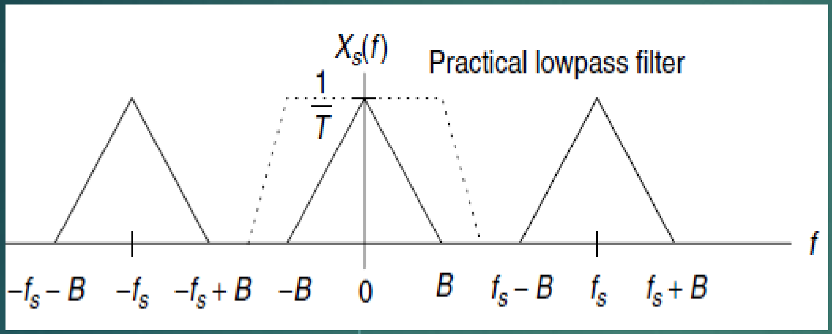
Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Signal Reconstruction

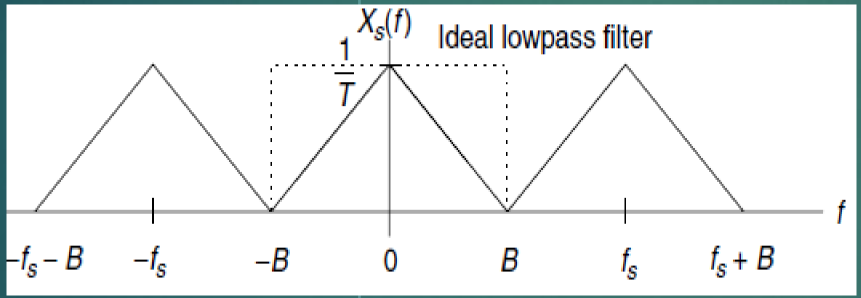


- The digitally processed data y(n) are converted to the ideal impulse train $y_s(t)$
- impulse has its amplitude proportional to digital output y(n), and two consecutive impulses are separated by a sampling period of T second.,
- the analog reconstruction filter is applied to the ideally recovered sampled signal y_s(t) to obtain the recovered analog signal.

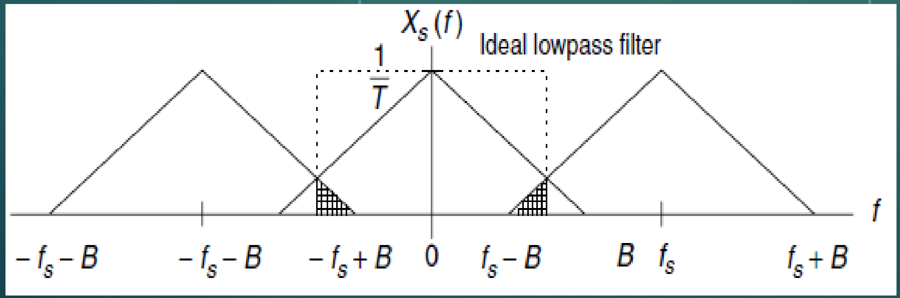
The possible three cases for the recovery of the original signal spectrum X(f).



$f_s > 2f_{max}$



$f_s = 2f_{max}$



$f_s < 2f_{max}$

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Example on Signal Recovery

Assuming that an analog signal given dye

X(t)=5cos (2π 2000t)+3cos(2π 3000t), for t≥ 0

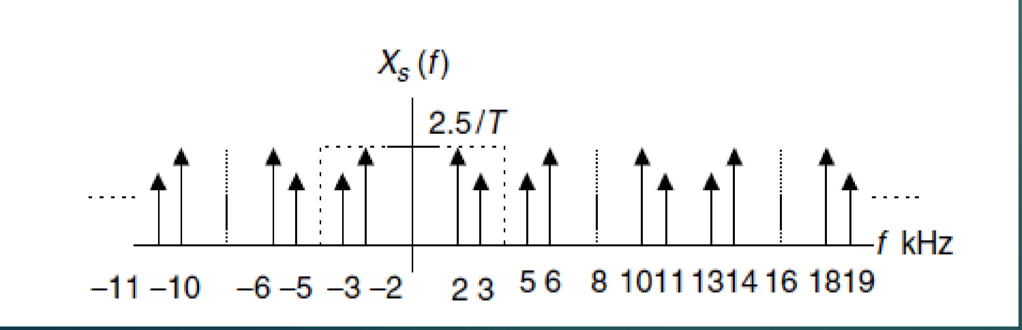
And it is sampled at the rate of 8kHz

a. Sketch the spectrum of the sampled signal up to 20 kHz
b. Sketch the recovered analog signal spectrum if an ideal lowpass filter with a cutoff frequency of 4kHz is used to filter the sampled signal (y(n)=x(n))to recover the original signal.

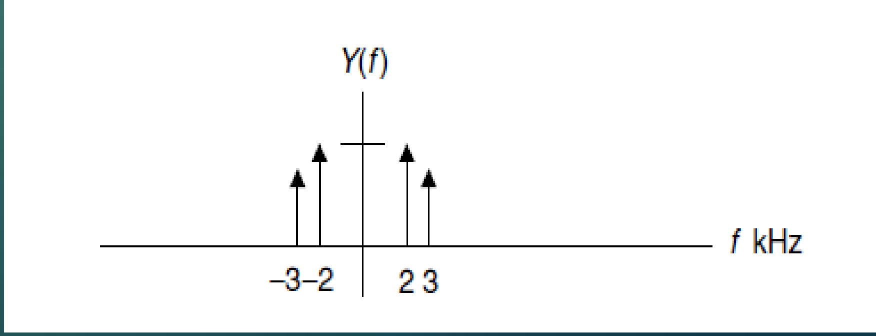
- Using Euler's identity

$$X(t)=\frac{3}{2}e^{-j2\pi\, 3000t}+\frac{5}{2}e^{-j2\pi\, 2000t}+\frac{5}{2}e^{j2\pi\, 2000t}+\frac{3}{2}e^{j2\pi\, 3000t}$$

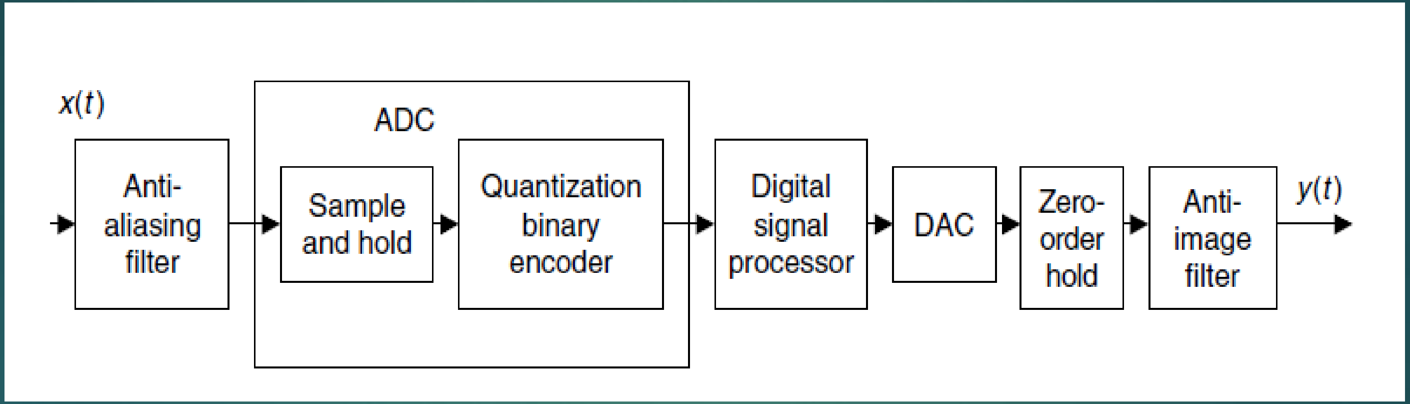
a. sampled signal



b. recovered signal

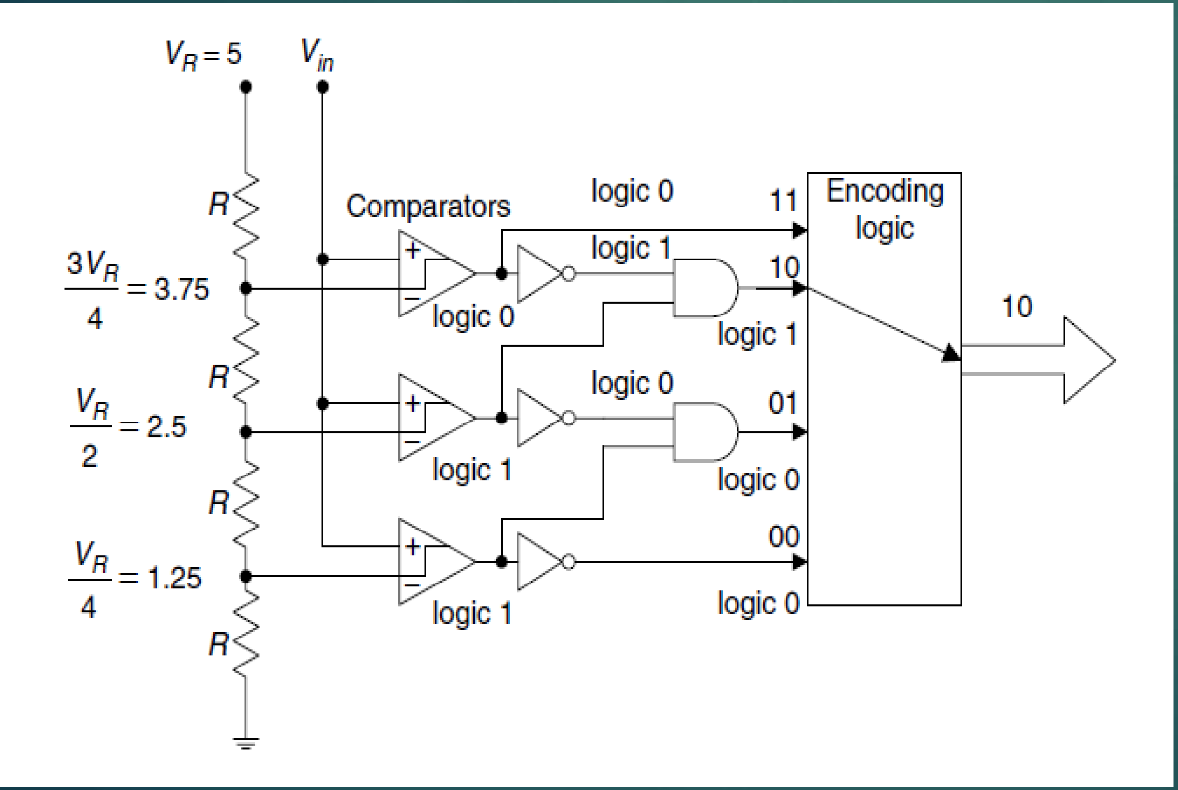


Analog-to-Digital Conversion, Digital-to-Analog Conversion, and Quantization



- The antialiasing filter is designed to block the frequency components beyond the folding frequency before the ADC operation, while the reconstruction filter is to block the frequency components beginning at the lower edge of the first image after the DAC.
- There are several ways to implement ADC. The most common ones are
- flash ADC,
- successive approximation ADC, and
- sigma-delta ADC.

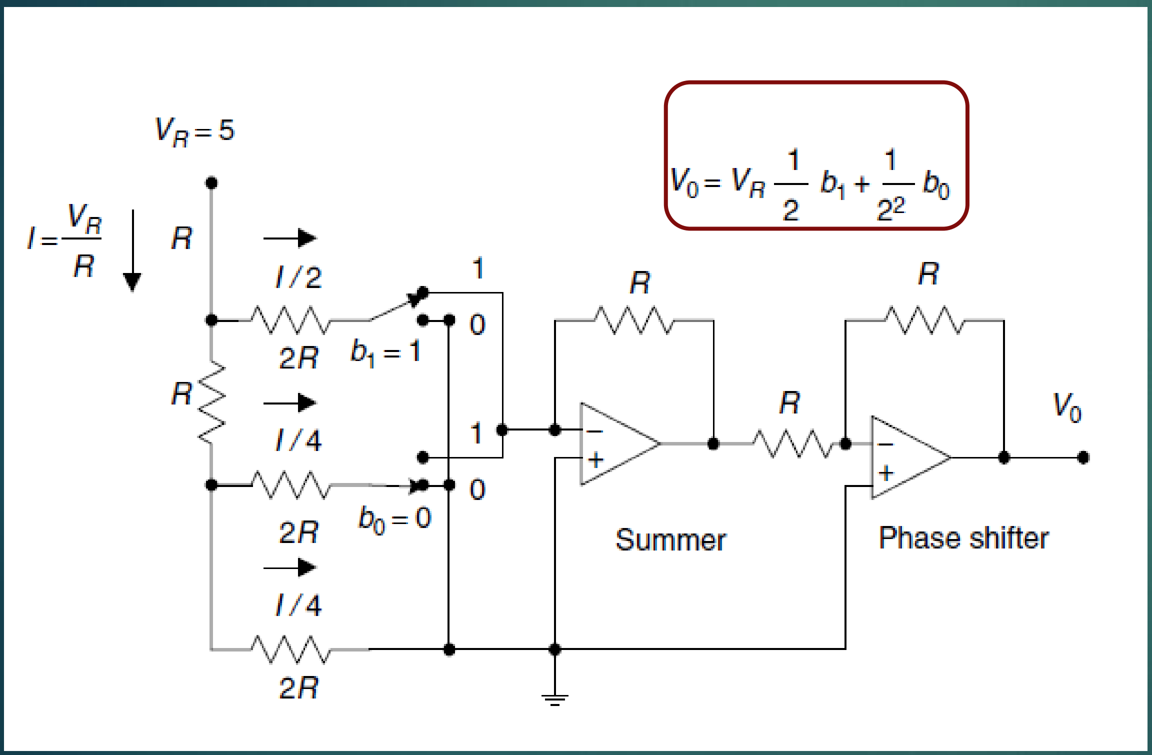
Analog-to-Digital Conversion, Digital-to-Analog Conversion, and Quantization



- the 2-bit flash ADC unit consists of a serial reference voltage created by the equal value resistors, a set of comparators, and logic units.

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Analog-to-Digital Conversion, Digital-to-Analog Conversion, and Quantization



- The DAC contains the R-2R ladder circuit
- a set of single-throw switches, a summer
- a phase shifter
- If a bit is logic 0, the switch connects a 2R resistor to ground
- If a bit is logic 1, the corresponding 2R resistor is connected to the branch to the input of the operational amplifier

Analog-to-Digital Conversion, Digital-to-Analog Conversion, and Quantization

- $V_R = 5, b_1 b_1 = 10$ the ADC output :

$V_0 = 5 \times \left(\frac{1}{2^1} \times 1 + \frac{1}{2^2} \times 0\right)$=2.5volts

- As we can see, the recovered voltage of V0 = 2.5 volts introduces voltage error as compared with Vin = 3, discussed in the ADC stage. $V_0 - V_{in} = 2.5 - 3 = -0,5V$
- Next, we focus on quantization development
- The notations and general rules for quantization are:

$$\Delta = \frac{(x_{max} - x_{\min})}{L}$$

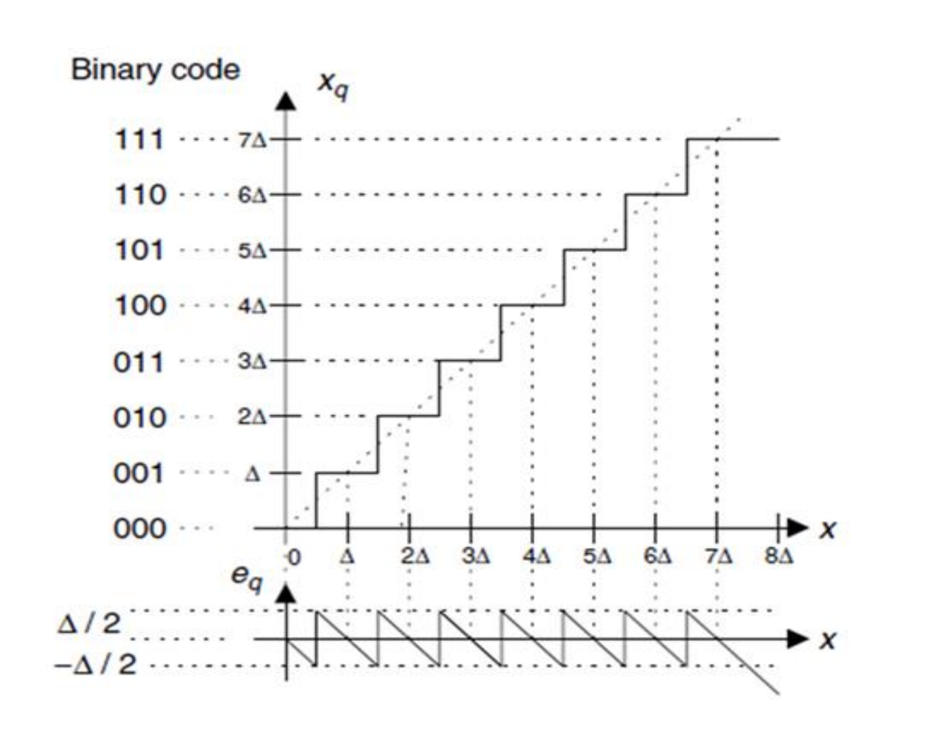
$$i = \text{round}\left(\frac{x - x_{min}}{\Delta}\right)$$

$$L = 2^m$$

$$x_q = x_{min} + i\Delta \text{ for } i = 0,1,\ldots,L-1$$

- x_{max} and x_{min} are the maximum and minimum values

Analog-to-Digital Conversion, Digital-to-Analog Conversion, and Quantization (Example)

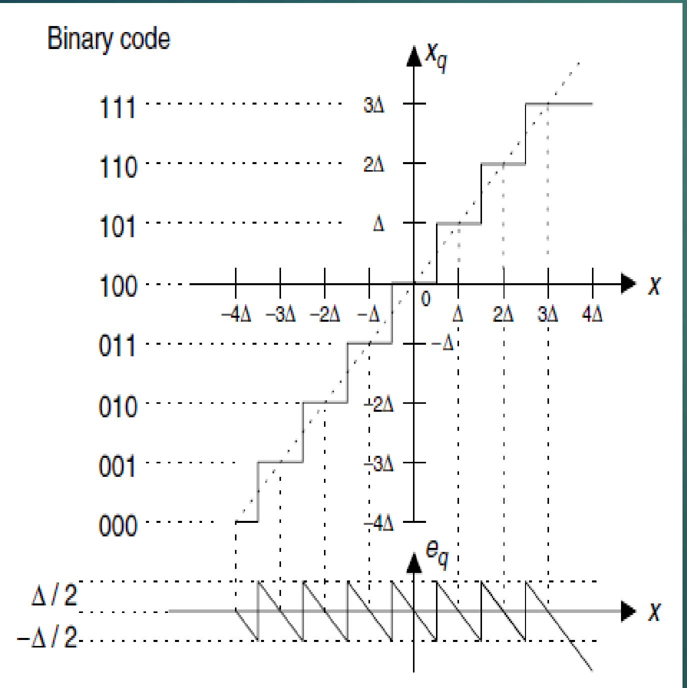


- $x_{min} = 0, x_{max} = 8\Delta$, m = 3
- x_q=0+iΔ,i=0,1,….,L-1
- L=2^3 = 8
- i is the integer corresponding to the 3-bit binary code

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Analog-to-Digital Conversion, Digital-to-Analog Conversion, and Quantization (Example)

where we have χ_{min} =-4Δ, x_{max} =4Δ, and m =3. The corresponding quantization table is given in Table



Binary Code	Quantization Level x_q (V)	Input Signal Subrange (V)
000	-4Δ	$-4\Delta \le x < -3.5\Delta$
001	-3Δ	$-3.5\Delta \le x < -2.5\Delta$
010	-2Δ	$-2.5\Delta \le x < -1.5\Delta$
011	$-\Delta$	$-1.5 \le x < -0.5\Delta$
100	0	$-0.5\Delta \le x < 0.5\Delta$
101	Δ	$0.5\Delta \le x < 1.5\Delta$
110	2Δ	$1.5\Delta \le x < 2.5\Delta$
111	3Δ	$2.5\Delta \le x < 3.5\Delta$

Analog-to-Digital Conversion, Digital-to-Analog Conversion, and Quantization (Example)

If the analog signal to be quantized is a sinusoidal waveform, that is

$$X(t)=a\sin(2\pi\times 1000t)$$

And if the bipolar quantizes use m bits , determine the SMR in terms of m bits.

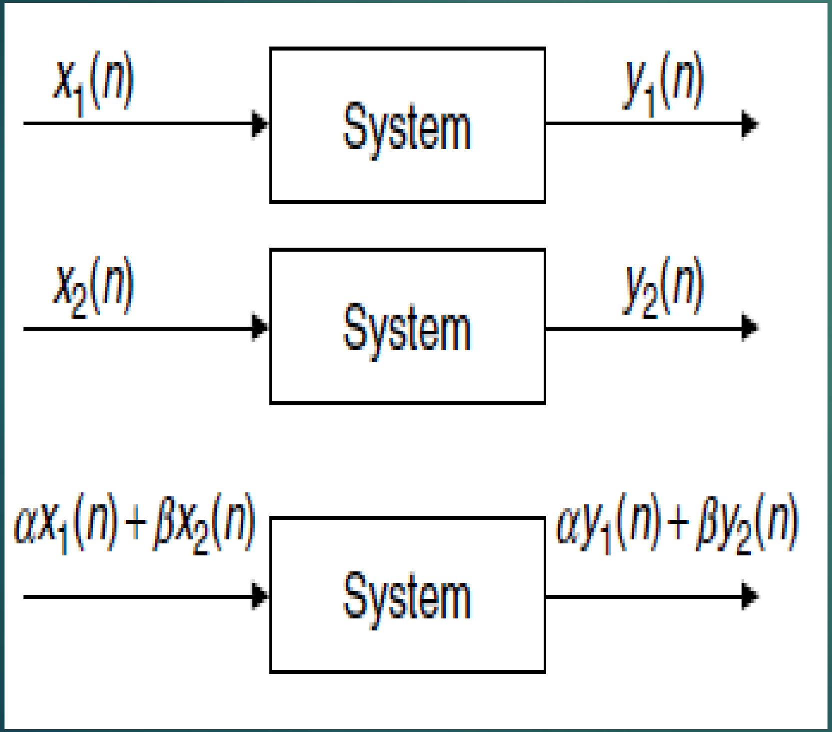
Solution:

a. Since x_{rms}=0,707A and Δ= 2A/2m

$SMR_{db} = 10,79 + 20\log\frac{0,707A}{2A/2^m}$ =10,79+20 $\log_{10}$ 0,707/2+20_m $\log_{10}$ 2

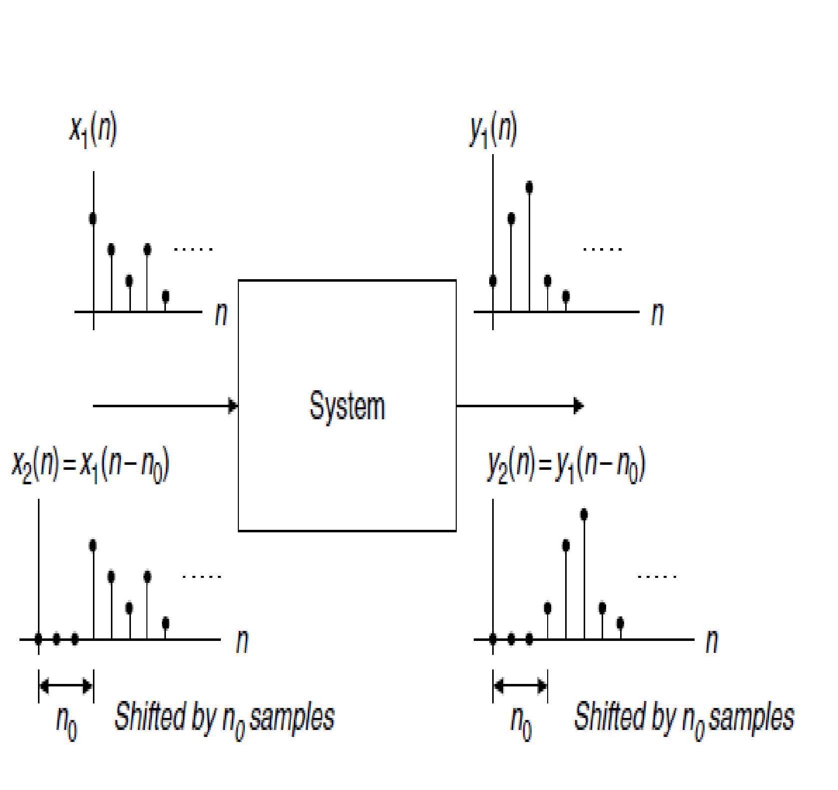
SMR_{db}= 1,76+6,02m db

Linear Time-Invariant, Causal Systems(Linearity)



- A linear system is illustrated in Figure
- using an input x_1(n), and x_2(n) is the system output using an input x_2(n).
- corresponding inputs : y(n)=a$y_1(n)$ + β$y_2(n)$
- where a and β are constants.

Time Invariance



- where y_1(n) is the system output for the input x_1(n). Let x_2(n) = x_1(n n_o) be the shifted version of x_1(n) by n_o samples
- The output y_2(n) obtained with the shifted input x_2(n) =x_1(n-n_o)

Causality

if a system output depends on the future input values, such as x(n + 1), x(n +2), . . . , the system is noncoastal. The noncoastal system cannot be realized in real time.

Example: Given the following linear system

a . y(n)= 0,5x(n)+2,5x(n-2)for n≥ 0

b. 0,25x(n-1)+0,5x(n+1)-0,4y(n-1) for n≥ 0 determine whether each is causal

Solution: a. for n≥ 0 the output y(n) depends on the current input x(n)and its past value x(n-2),the system is causal.

b. For n≥ 0,the output y(n) depends on the current input x(n) and its past value x(n+2) ,the system is noncausal.

Fourier Series and Fourier Transform

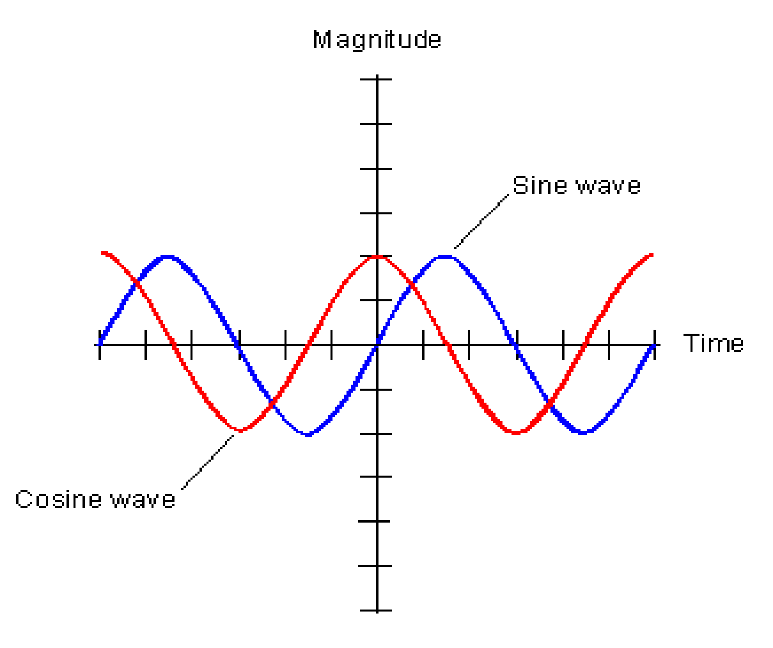
Periodic signals, such as the square wave, rectangular wave, triangular wave, sinusoid, sawtooth Wave etc, can be analyzed in frequency domain with the help of the Fourier series expansion.

According to Fourier theory, a periodic signal can be represented by a Fourier series that contains the sum of a series of sine and/or cosine functions (harmonics) plus a direct-current (dc) term.

There are three forms of Fourier series:

- **sine-cosine**
- **amplitude-phase**
- **complex exponential**

Sine-Cosine Form



Sine-cosine form is given by:

$$x_{(t)} = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{i=0}^{\infty} b_n \sin(n\omega_0 t)$$

where $\omega_{0=} {}^{2\pi}/_{T_0}$ is the fundamental angular frequency in radians/second, while the fundamental frequency, in terms of Hz, is: $f_0 = {}^{1}/_{T_o}$.

Fourier coefficients :

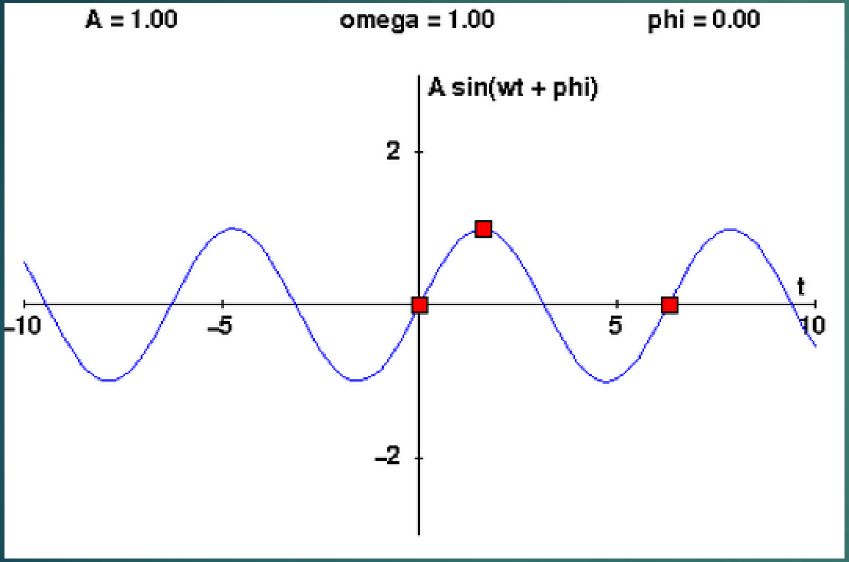
$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)dt$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)dt$$

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Amplitude-Phase Form



The amplitude-phase form:

$$x(t)= A_o + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \Phi_n)$$

DC term is the same: $A_0 = a_0$.

The amplitude and phase of n^{th}-harmonic are given by:

$$A_n = \sqrt{{a_n}^2 + {b_n}^2}$$

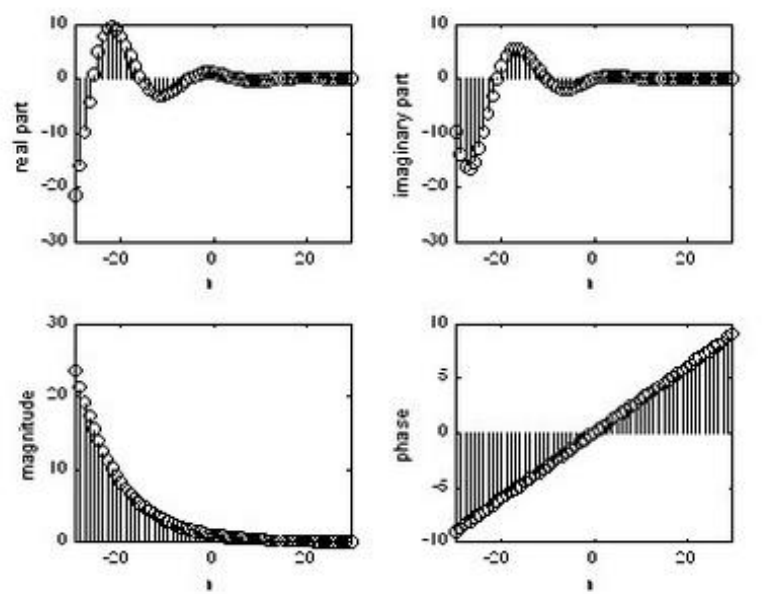
$$\Phi_n = \tan^{-1}\frac{-b_n}{a_n}$$

The amplitude-phase form provides very useful information for spectral analysis.

Technological Educational Institute of Crete
Anna Daskalaki| Crete | Greece
d_anna-13-ntora@windowslive.com

Complex Exponential Form

Example: $x(n)=0.9^n e^{j0.3n}$



Euler's formula is given by:

$$e^{\pm jx} = \cos(x) \pm j\sin(x)$$

can be written as two separate forms:

$$\cos(x) = \frac{e^{jx}+e^{-jx}}{2}$$

$$\sin(x) = \frac{e^{jx}-e^{-jx}}{2j}$$

the complex exponential form is expressed as:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

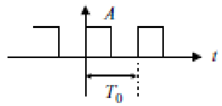
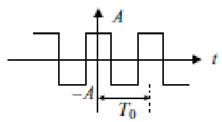
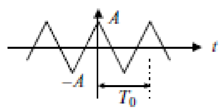
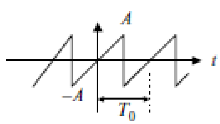
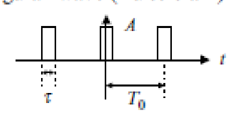
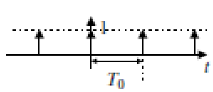
$$c_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}\,\mathrm{d}t$$

$$c_0 = a_0 \qquad c_n = \frac{a_n - jb_n}{2}$$

$$c_{-n} = \overline{c_n} = \frac{a_n + jb_n}{2}$$

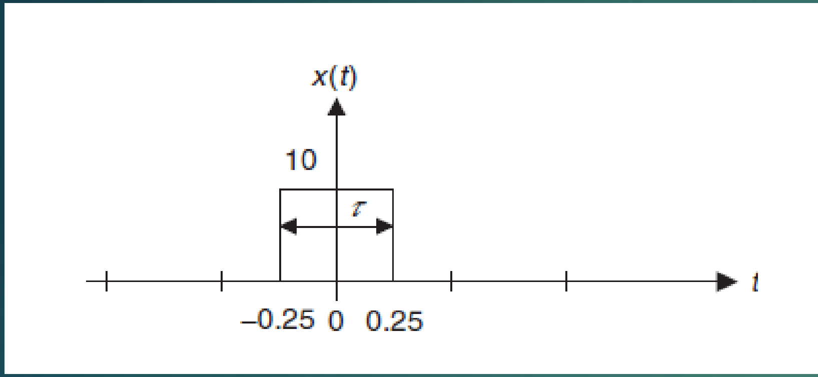
$$c_n = |c_n|\angle\varphi_n$$

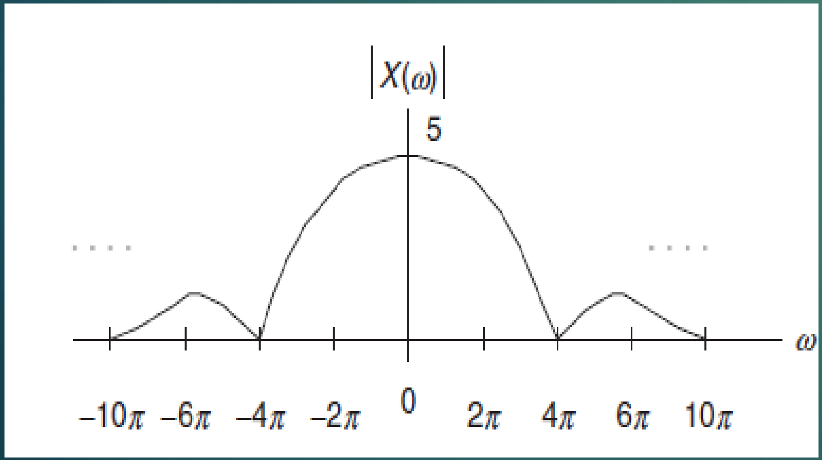
Fourier series expansions for some common signals

Time domain	sine-cosine form	complex exponential form
Positive square wave 	$x(t) = \frac{A}{2} + \frac{2A}{\pi}\left(\sin \omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \frac{1}{7}\sin 7\omega_0 t + \cdots\right)$	$x(t) = \cdots - \frac{A}{j3\pi}e^{-j3\omega_0 t} - \frac{A}{j\pi}e^{-j\omega_0 t} + \frac{A}{2} + \frac{A}{j\pi}e^{j\omega_0 t} + \frac{A}{j3\pi}e^{j3\omega_0 t} + \frac{A}{j5\pi}e^{j5\omega_0 t} + \cdots)$
Square wave 	$x(t) = \frac{4A}{\pi}\left(\cos \omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \frac{1}{7}\cos 7\omega_0 t + \cdots\right)$	$x(t) = \frac{2A}{\pi}\left(\cdots + \frac{1}{5}e^{-j5\omega_0 t} - \frac{1}{3}e^{-j3\omega_0 t} + e^{-j\omega_0 t} + e^{j\omega_0 t} - \frac{1}{3}e^{j3\omega_0 t} + \frac{1}{5}e^{j5\omega_0 t} - \cdots\right)$
Triangular wave 	$x(t) = \frac{8A}{\pi^2}\left(\cos \omega_0 t + \frac{1}{9}\cos 3\omega_0 t + \frac{1}{25}\cos 5\omega_0 t + \frac{1}{49}\cos 7\omega_0 t + \cdots\right)$	$x(t) = \frac{4A}{\pi^2}\left(\cdots + \frac{1}{25}e^{j5\omega_0 t} + \frac{1}{9}e^{-j3\omega_0 t} + e^{-j\omega_0 t} + e^{j\omega_0 t} + \frac{1}{9}e^{j3\omega_0 t} + \frac{1}{25}e^{j5\omega_0 t} + \cdots\right)$
Sawtooth wave 	$x(t) = \frac{2A}{\pi}\left(\sin \omega_0 t - \frac{1}{2}\sin 2\omega_0 t + \frac{1}{3}\sin 3\omega_0 t - \frac{1}{4}\sin 4\omega_0 t + \cdots\right)$	$x(t) = \frac{A}{j\pi}\left(\cdots - \frac{1}{3}e^{-j3\omega_0 t} + \frac{1}{2}e^{-j2\omega_0 t} - e^{-j\omega_0 t} + e^{j\omega_0 t} - \frac{1}{2}e^{j2\omega_0 t} + \frac{1}{3}e^{j3\omega_0 t} + \cdots\right)$
Rectangular wave (Pulse train)  Duty cycle $= d = \frac{\tau}{T_0}$	$x(t) = Ad + 2Ad\left(\frac{\sin \pi d}{\pi d}\right)\cos \omega_0 t + 2Ad\left(\frac{\sin 2\pi d}{2\pi d}\right)\cos 2\omega_0 t + 2Ad\left(\frac{\sin 3\pi d}{3\pi d}\right)\cos 3\omega_0 t + \cdots$	$x(t) = \cdots + Ad\left(\frac{\sin \pi d}{\pi d}\right)e^{-j\omega_0 t} + Ad + Ad\left(\frac{\sin \pi d}{\pi d}\right)e^{j\omega_0 t} + Ad\left(\frac{\sin 2\pi d}{2\pi d}\right)e^{j2\omega_0 t} + Ad\left(\frac{\sin 3\pi d}{3\pi d}\right)e^{j3\omega_0 t} + \cdots$
Ideal impulse train 	$x(t) = \frac{1}{T_0} + \frac{2}{T_0}(\cos \omega_0 t + \cos 2\omega_0 t + \cos 3\omega_0 t + \cos 4\omega_0 t + \cdots)$	$x(t) = \frac{1}{T_0}(\cdots + e^{-j3\omega_0 t} + e^{-j2\omega_0 t} + e^{-j\omega_0 t} + 1 + e^{j\omega_0 t} + e^{j2\omega_0 t} + e^{j3\omega_0 t} + \cdots)$

Anna Daskalaki | d_anna-13-ntora@windowslive.com

Fourier Transform





Fourier transform is a mathematical function that provides the frequency spectral analysis for a non-periodic signal. The Fourier transform pair is defined as:

$$X(\omega) = \int_{-\infty}^{\infty} \boldsymbol{x(t) e^{-j\omega t} dt}$$

Inverse Fourier transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\boldsymbol{\omega})\, \boldsymbol{e^{j\omega t}} d\omega$$

The spectrum is a complex function that can be further written as:

$$X(\omega) = |X(\omega)| \ \angle\Phi(\omega)$$

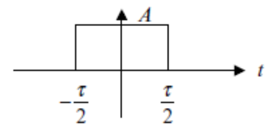
where $-\infty < \omega < \infty$, $|X(\omega)|$ is the continuous amplitude spectrum, while $\angle\Phi(\omega)$ designates the continuous phase spectrum.

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Fourier transform for some common signals

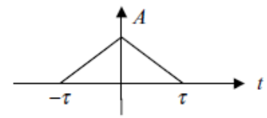
Time domain — **Fourier Spectrum**

Rectangular pulse



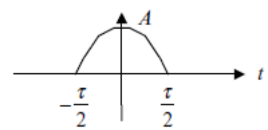
$$X(f) = A\tau \frac{\sin \pi f \tau}{\pi f \tau}$$

Triangular pulse



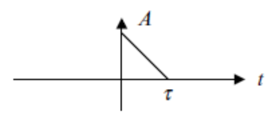
$$X(f) = A\tau \left(\frac{\sin \pi f \tau}{\pi f \tau}\right)^2$$

Cosine pulse



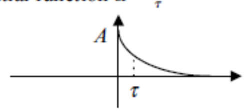
$$X(f) = \frac{2A\tau}{\pi}\left(\frac{\cos \pi f \tau}{1-4f^2\tau^2}\right)$$

Sawtooth pulse



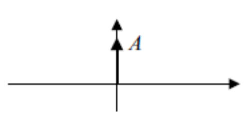
$$X(f) = \frac{jA}{2\pi f}\left(\frac{\sin \pi f \tau}{\pi f \tau} e^{-j\pi f \tau} - 1\right)$$

Exponential function $\alpha = \frac{1}{\tau}$



$$X(f) = \frac{A}{\alpha + j2\pi f}$$

Impulse function



$$X(f) = A$$

Properties of Fourier Transform

Time Function	Fourier Transform
$\alpha x_1(t) + \beta x_2(t)$	$\alpha X_1(f) + \beta X_2(f)$
$\frac{dx(t)}{dt}$	$j2\pi f X(f)$
$\int_{-\infty}^{t} x(t)dt$	$\frac{X(f)}{j2\pi f}$
$x(t-\tau)$	$e^{-j2\pi f\tau} X(f)$
$e^{j2\pi f_0 t} x(t)$	$X(f-f_0)$
$x(at)$	$\frac{1}{a} X\left(\frac{f}{a}\right)$

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Exercise:

Let ***x(t)*** be an exponential function given by:

$$x(t) = 10e^{-2t}u(t) = \begin{cases} 10e^{-2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Find its Fourier transform.

Solution:

$$X(\omega) = \int_0^{\infty} 10e^{-2t}u(t)\, e^{-j\omega t} dt = \int_0^{\infty} 10e^{-(2+j\omega)t} dt$$

$$= \left.\frac{10e^{-(2+j\omega)t}}{-(2+j\omega)}\right|_0^{\infty} = \frac{10}{2+j\omega}$$

$$X(\omega) = \frac{10}{\sqrt{2^2+\omega^2}} \angle -tan^{-1}\left(\frac{\omega}{2}\right)$$

For $\omega = 2\pi f$

$$X(\omega) = \frac{10}{2+j2\pi f} = \frac{10}{\sqrt{2^2+(2\pi f)^2}} \angle -tan^{-1}(\pi f)$$

Exercise:

- Find the Fourier transforms of the following functions
- a. $\boldsymbol{x_{(t)} = \delta_{(t)}}$ where $\delta_{(t)}$ in an impulse function defined
  - $\delta_{(t)} = \begin{cases} \neq \mathbf{0} & \boldsymbol{t = 0} \\ \mathbf{0} & \boldsymbol{elsewhere} \end{cases}$
  - with a property given as
  - $\int_{-\infty}^{\infty} \boldsymbol{f_{(t)}\delta(t-\tau)dt = f_{(\tau)}}$
- b. $\boldsymbol{x_{(t)} = \delta(t-\tau)}$
- Solution:

a. $\boldsymbol{X_{(\omega)}} = \int_{-\infty}^{\infty} \delta_{(t)} \boldsymbol{e^{-j\omega t} dt} = e^{-j\omega t}\Big|_{\boldsymbol{t=0}} = \mathbf{1}$

b. $\boldsymbol{X_{(\omega)}} = \int_{-\infty}^{\infty} \delta(t-\tau) e^{-j\omega t} \boldsymbol{dt} = e^{-j\omega t}\Big|_{\boldsymbol{t=\tau}} = e^{-\boldsymbol{j\omega\tau}}$

Laplace Transform

Laplace transform plays an important role in analysis of continuous signals and systems. We define Laplace transform pairs as:

$$X(s) = L\{x(t)\} = \int_0^{\infty} x(t) e^{-st} dt$$

$$x(t) = L^{-1}\{X(s)\} = \frac{1}{2\pi j} \int_{\gamma - j\infty}^{\gamma + j\infty} X(s) e^{st} ds$$

Notice that the symbol L{ } denotes the forward Laplace operation, while the symbol L^{-1}\{ \} indicates the inverse Laplace operation.

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Properties of Laplace Transform

Time domain	's' domain
$af(t)+bg(t)$	$aF(s)+bG(s)$
$tf(t)$	$-F'(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$s^nF(s)-\sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0)$
$\frac{1}{t}f(t)$	$\int_s^\infty F(\sigma)\,d\sigma$
$\int_0^t f(\tau)\,d\tau=(u*f)(t)$	$\frac{1}{s}F(s)$
$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
$e^{at}f(t)$	$F(s-a)$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$f(t)g(t)$	$\frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT}F(\sigma)G(s-\sigma)\,d\sigma$
$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)\cdot G(s)$
$f^*(t)$	$F^*(s^*)$
$f(t)\star g(t)$	$F^*(-s^*)\cdot G(s)$
$f(t)$ *periodic function*	$\frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)\,dt$

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Reference Table of Laplace Transform

$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1	$\frac{1}{s}$
$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$
$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$
$t^n e^{at},\quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$
$u_c(t)=u(t-c)$ Heaviside Function	$\frac{e^{-cs}}{s}$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$
$\frac{1}{t}f(t)$	$\int_s^{\infty}F(u)\,du$
$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$

$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
e^{at}	$\frac{1}{s-a}$
$t^p,\ p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
$\delta(t-c)$ Dirac Delta Function	e^{-cs}
$u_c(t)g(t)$	$e^{-cs}\mathfrak{L}\{g(t+c)\}$
$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
$f(t+T)=f(t)$	$\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$

Transfer Function

A linear analog system can be described using the Laplace transfer function. The transfer function, relating the input and output of the linear system, is defined as a ratio of the Laplace response of the system to the Laplace input given by:

$$H(s)=\frac{Y(s)}{X(s)}$$

If X(s)=1, the output of the linear system due to the impulse function is:

$$Y(s)=H(s)X(s)=H(s)$$

Therefore, the response in time domain is called the ***impulse response of the system*** and can be expressed:

$$h(t)=L^{-1}\{H(s)\}$$

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Transfer Function example

Canside a linear system y(t)=0,5u(t)-0,5$e^{-10t}u(t)$ designate the system imput and system output respectively.

a. Derive the transfer function and the impulse response of the system.

Solution:

a. Tacking the Laplace transform on both sides the differential equation yields

$$\mathrm{L}\left\{\frac{d_y(t)}{d_t}\right\} + \mathrm{L}\{10y(t)\} = L\{x(t)\}$$

Applying the differential property and substituting the initial condition, we have Y(s)(s +10)=X(s)

Thus, the transfer function is given by H(s) $\frac{Y(s)}{X(s)} = \frac{1}{s+10}$

The impuls response can be found by taking the inverse Laplace transform as

$$\mathrm{h(t)} = L^{-1}\left\{\frac{1}{s+10}\right\} = e^{-10t}\mathrm{u(t)}$$

Poles, Zeros, and Stability

To study system behavior, the transfer function is written in the general form:

$$\mathrm{H(s)} = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots a_0}$$

Given a system transfer function, the poles [roots of *D(s)*] and zeros [roots of *N(s)*] can be found. Notice that zeros and poles can be real or complex numbers.

Stability of the system is determined by the following rules:

- The linear system is <u>stable</u> if the rightmost pole(s) is(are) on the left-hand half plane (LHHP) on the s-plane.
- The linear system is <u>marginally stable</u> if the rightmost pole(s) is(are) simple (first order) on the jω axis, including the origin on the s-plane.
- The linear system is <u>unstable</u> if the rightmost pole(s) is(are) on the righthand half plane (RHHP) of the s-plane or if the rightmost pole(s) is(are) multiple order on the jω axis on the s-plane.
- Zeros do not affect the system stability.

Poles, Zeros, and Stability example

Determine whether each of following transfer function is stable marginally stable ,or unstable

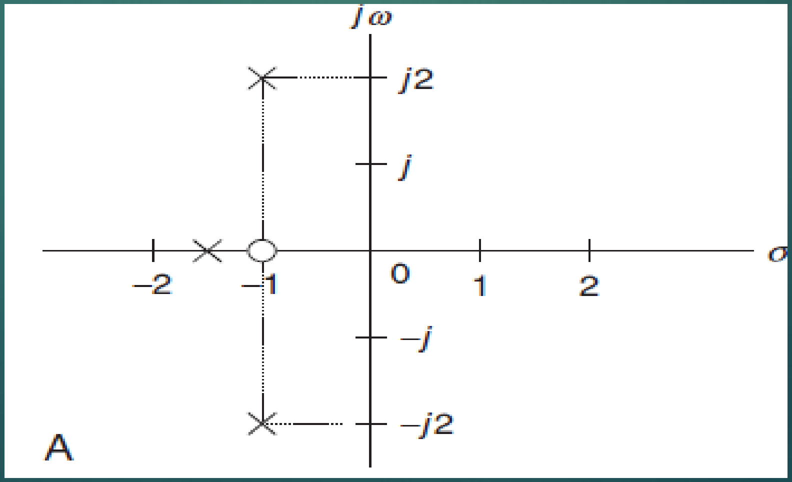
1. $H(s)=\frac{s+1}{(s+1,5)(s^2+2s+5)}$

Solution :

1. A zero is found to be s= -1

The poles are calculated as s = -1,5 ,s = -1+j2 , s = -1 –j2

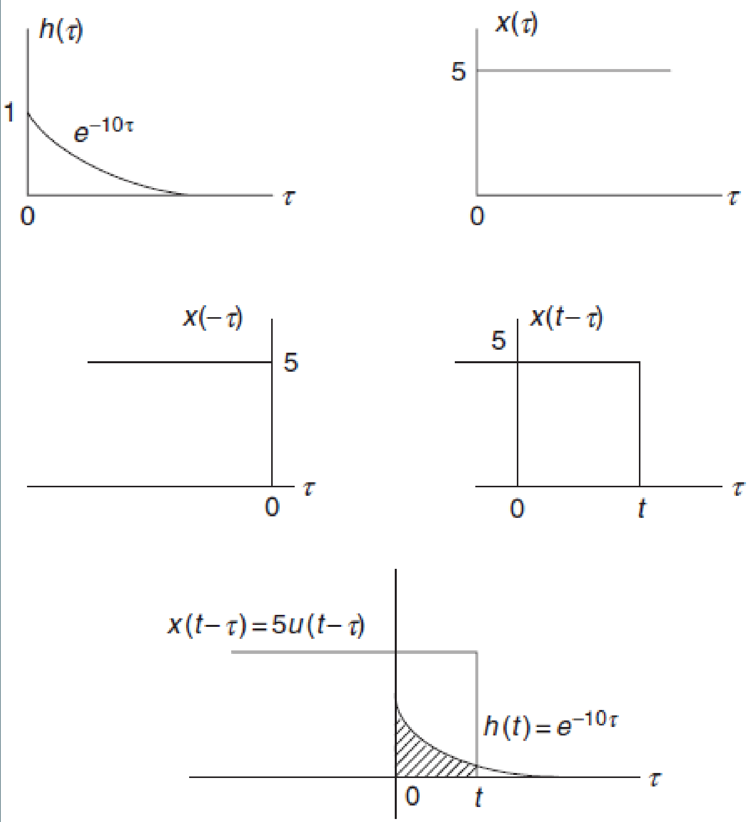
The poles-zero plot is show in figure .Since all the poles are located an the LHHP, the system is stable.



Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Convolution

$$y(t) = 0.5u(t) - 0.5e^{-10t}u(t).$$



In the Laplace domain, the system output is the product of the Laplace input and the transfer function:

$$Y(s)=H(s)\cdot X(s)$$

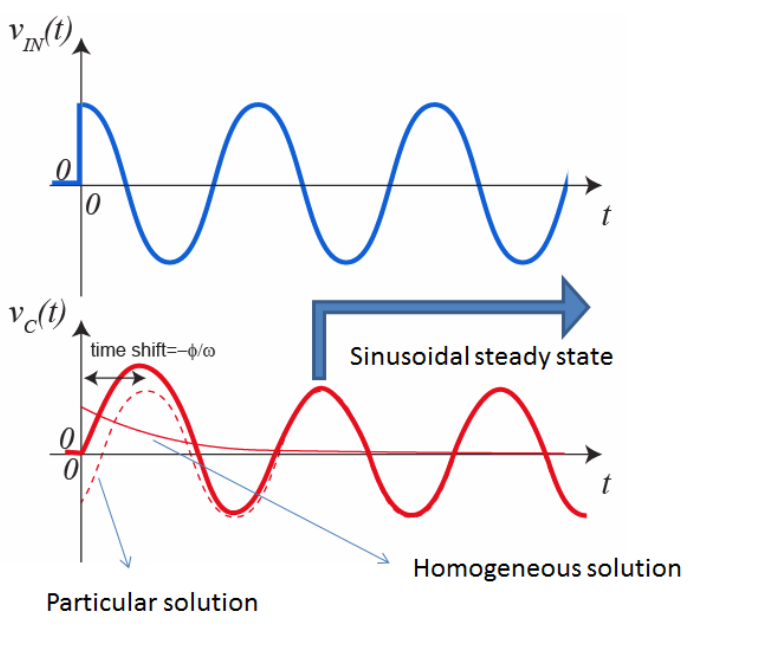
But in time domain, the system output is given as:

$$y(t) = h(t) * x(t)$$

The linear convolution is further expressed as:

$$y(t) = \int_0^{\infty} h(\tau)x(t-\tau)\,d\tau$$

Sinusoidal Steady-State Response



For linear analog systems, if the input is a sinusoid of radian frequency ω, the steady-state response of the system will also be a sinusoid of the same frequency and the transfer function is called the **steady-state transfer function**:

$$H(j\omega) = H(s)|s = j\omega$$

Thus for a system in sinusoidal steady state:

$$Y(j\omega) = H(j\omega) \cdot X(j\omega)$$

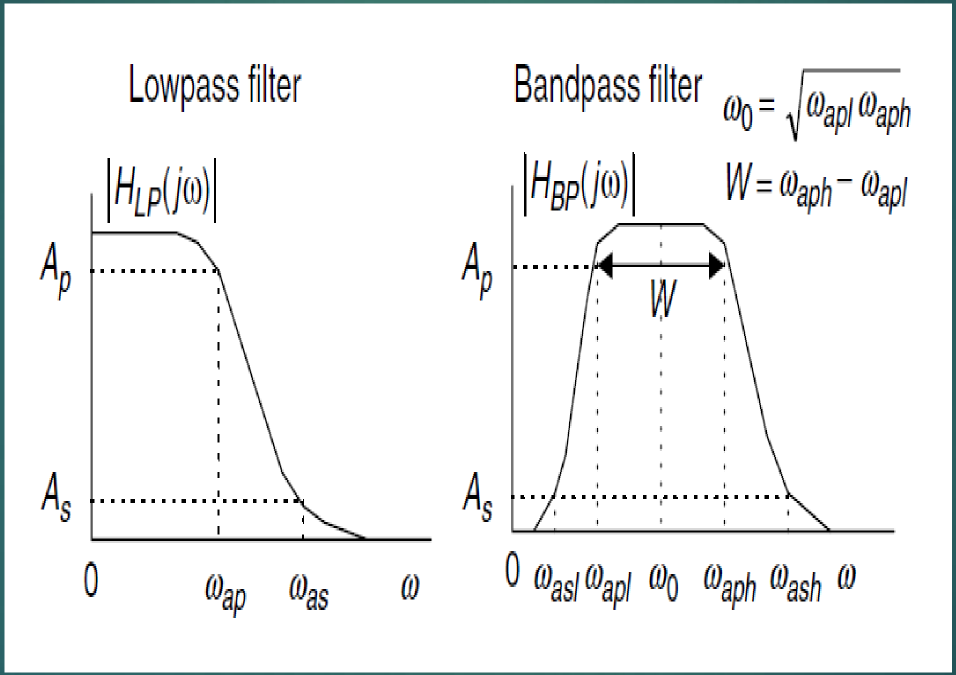
The complex steady-state transfer function, can be written in phasor form:

$$H(j\omega) = A(j\omega)\angle\beta(\omega)$$

where $A(j\omega) = |H(j\omega)|$ and $\beta(\omega)$ is the phase response of the system.

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Specifications for analog low pass and band pass filters.



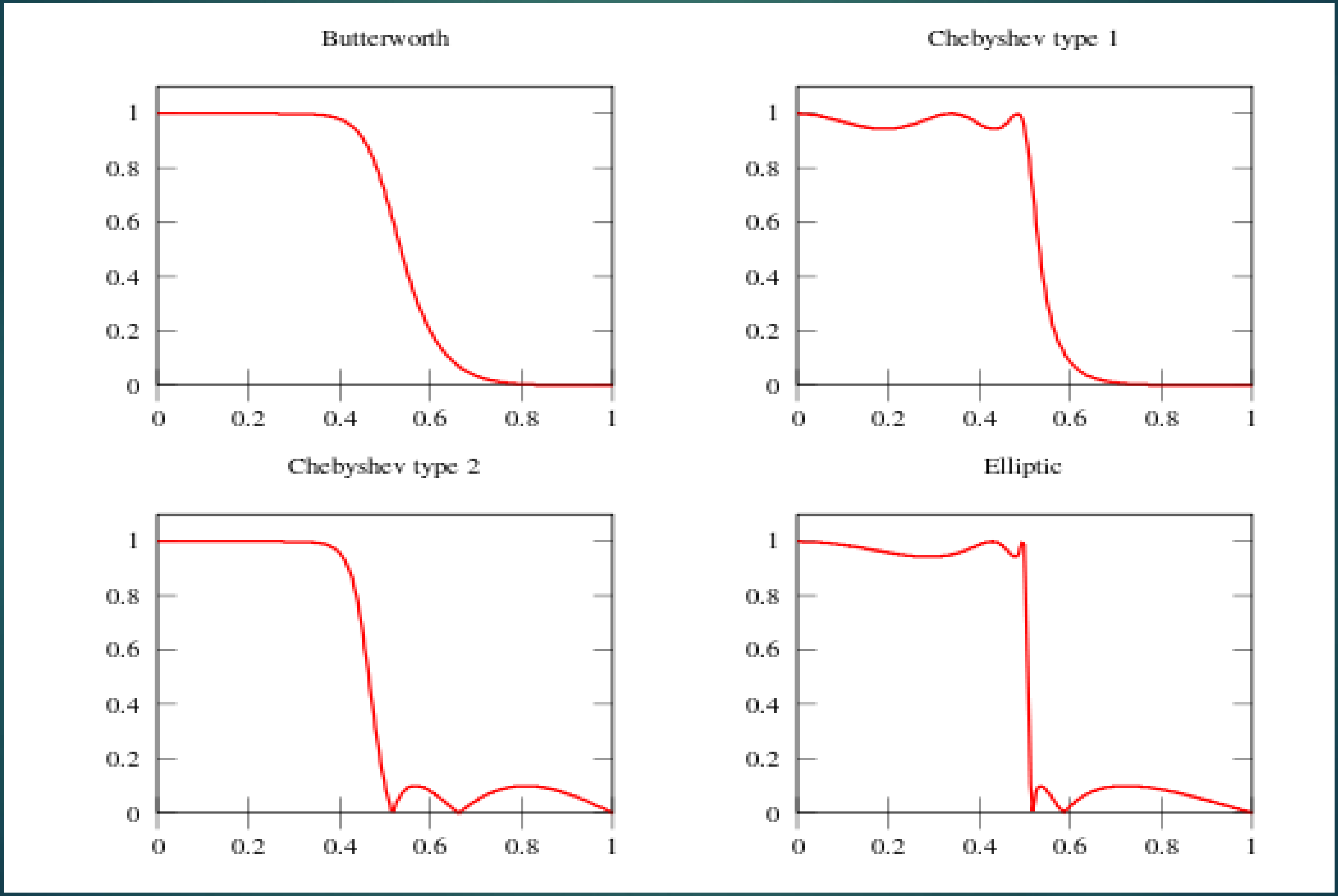
- ✓ Frequency edge notations for analog low pass and band pass filters. The notations for analog high pass and band stop filters can be defined correspondingly.

Low pass Prototype Function and Its Order

- BLT design requires obtaining the analog filter with prewrapped frequency specifications.
- These analog filter design requirements include the ripple specification at the passband frequency edge
- the attenuation specification at the ,stopband frequency edge
- type of low pass prototype, and its order

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Function



Anna Daskalaki | d_anna-13-ntora@windowslive.com

Butterworth low pass

- The magnitude response function of the Butterworth low pass prototype with an order of n is shown

$$|H_p|(\nu)=\frac{1}{\sqrt{1+\varepsilon^2\nu^{2n}}}$$

- the given passband ripple A_p dB at the normalized passband frequency edge ν_p =1, and the stopband attenuation A_sdB at the normalized stopband frequency

$$A_p dB = -20\ \log_{10}\left(\frac{1}{\sqrt{1+\varepsilon^2}}\right)$$

$$A_s dB = -20\ \log_{10}\left(\frac{1}{\sqrt{1+\varepsilon^2 {\nu_s}^{2n}}}\right)$$

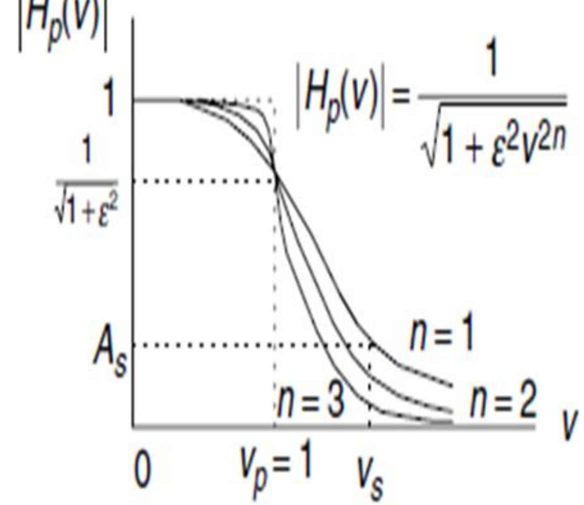
- Low pass prototype order as : $\varepsilon^2 = 10^{0,1A_p} - 1$

$$n \geq \frac{\log_{10}\left(\frac{10^{0,1A_s}-1}{\varepsilon^2}\right)}{[2\ \log_{10}(\nu_s)]},$$

- ε is the absolute ripple specification

Anna Daskalaki | d_anna-13-ntora@windowslive.com

Butterworth low pass



Normalized Butterworth magnitude response function.

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Table 1 lists the Butterworth

- ✓ prototype functions with 3 dB passband ripple specification

transfer functions ($\varepsilon = 1$)

n	$H_P(s)$
1	$\frac{1}{s+1}$
2	$\frac{1}{s^2+1.4142s+1}$
3	$\frac{1}{s^3+2s^2+2s+1}$
4	$\frac{1}{s^4+2.6131s^3+3.4142s^2+2.6131s+1}$
5	$\frac{1}{s^5+3.2361s^4+5.2361s^3+5.2361s^2+3.2361s+1}$
6	$\frac{1}{s^6+3.8637s^5+7.4641s^4+9.1416s^3+7.4641s^2+3.8637s+1}$

Normalized Butterworth Function

- The normalized Butterworth squared magnitude function

$$|P_n(\omega)|^2=\frac{1}{1+\varepsilon^2(\omega)^{2n}}$$

- n is the order and « is the specified ripple on filter passband.
- specified ripple in dB $\varepsilon_{\mathrm{dB}} = 20\log_{10}(\sqrt{1+\varepsilon^2})$ dB .
- To develop the transfer function $P_n(s)$ we first let s=jω and then substitute $\omega^2 = -s^2$.

$$P_n(s)P_n(-s) = \frac{1}{1+\varepsilon^2(-s^2)^n} \quad (1)$$

- (1):has 2n poles, $P_n(s)$ has n poles on the left-hand half plane (LHHP) on the s-plane, while $P_n(-s)$ has n poles on the right-hand half plane (RHHP) on the s-plane.
- Solving for poles leads to $(-1)^n s^{2n} = -{}^1/_{\varepsilon^2}$.

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Normalized Butterworth Function

- and the corresponding poles are solved as
  - $P_k = \varepsilon^{-1/n} e^{j\frac{2\pi k}{2n}} = \varepsilon^{-1/n}[\cos {}^{2\pi k}/_{2n} + j\sin {}^{2\pi k}/_{2n}]$
- k=1,2,....2n, r= $\varepsilon^{-1/n}$, $\theta_\kappa = 2\pi k/(2n)$ for k= 0,1,....,2n-1
- and from a factor from the real pole (s + r), it follows that
  - $P_n(s) = \frac{K}{(s+r)\prod_{k=1}^{(n-1)/2}\left(s^2+(2r\cos(\theta_k)\right)s+r^2}$
- θ_k=2πk/(2n) for k=1,....,(n-1)/2
- K=$r^n = 1/\varepsilon$
- When n is an even number, we can identify the poles on the LHHP as
  - $p_k = -r\cos(\theta_k)$+jrsin (θ_k),k=1,.....,n/2 - 1

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Normalized Butterworth Function Example 1

- Complete the normalized Butterworth transfer function for the following specifications

1. Ripple=3db
2. N=2

Solution:

- n/2=1
- $\boldsymbol{\theta_k = \frac{2\pi\times0+\pi}{2\times2} = 0,25\pi}$
- $\boldsymbol{\varepsilon^2 = 10^{0,1\times3} - 1}$
- Applying equation leads to

$$\boldsymbol{P_2(s) = \frac{1}{s^2+2\times1\times cos(0,25\pi)s+1^2} = \frac{1}{s^2+1,414s+1}}$$

Anna Daskalaki | d_anna-13-ntora@windowslive.com

Normalized Chebyshev magnitude response function

- where the magnitude response versus the normalized frequency v is given by

$$|H_p|(v)=\frac{1}{\sqrt{1+\varepsilon^2 C_n{}^2(v)}}$$

$$C_n(v_s)=\cosh[n\cosh^{-1}(v_s)]$$

$$\cosh^{-1}(v_s)=\ln\left(\ln v_s+\sqrt{v_s{}^2-1}\right)$$

- As shown in Figure 8.14, the magnitude response for the Chebyshev low pass prototype with the order of an odd number begins with the filter DC gain of 1.

$$A_p dB=-20\ \log_{10}\left(\frac{1}{\sqrt{1+\varepsilon^2}}\right)$$

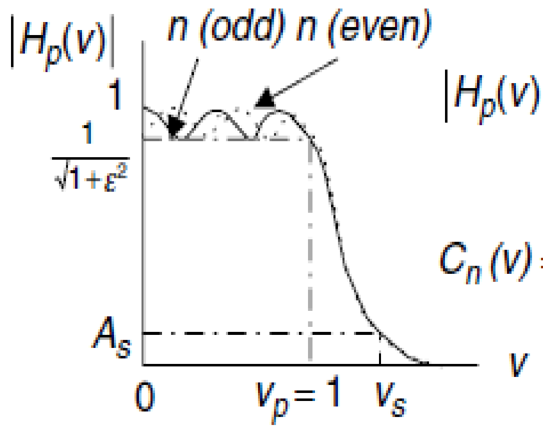
$$A_s dB=-20\ \log_{10}\left(\frac{1}{\sqrt{1+\varepsilon^2 C_n{}^2(v_s)}}\right)$$

$$\varepsilon^2=10^{0,1A_p}-1$$

$$n\geq\frac{\cosh^{-1}\left[\left(\frac{10^{0,1A_s-1}}{\varepsilon^2}\right)^{0,5}\right]}{\cosh^{-1}(v_s)}$$, $\cosh^{-1}(x)=\ln(x+\sqrt{x^2-1})$, ε is the absolute ripple parameter

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Normalized Chebyshev magnitude response function



$$|H_p(v)| = \frac{1}{\sqrt{1+\varepsilon^2 C_n^2(v)}}$$

$$C_n(v) = \begin{cases} \cos\left[n\cos^{-1}(v)\right] & v \le 1 \\ \cosh\left[n\cosh^{-1}(v)\right] & v > 1 \end{cases}$$

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Tables 2,3 and contain the Chebyshev prototype functions

- ✓ with 1 dB and 0.5 dB passband ripple specifications, respectively
- ✓ Other low pass prototypes with different ripple specifications
- ✓ order can be computed using the methods described
- ✓ The Chebyshev type II filter design can be found in Proakis and Manolakis (1996) and Porat (1997).

functions with 0.5 dB ripple ($\varepsilon = 0.3493$)

n	$H_P(s)$
1	$\frac{2.8628}{s+2.8628}$
2	$\frac{1.4314}{s^2+1.4256s+1.5162}$
3	$\frac{0.7157}{s^3+1.2529s^2+1.5349s+0.7157}$
4	$\frac{0.3579}{s^4+1.1974s^3+1.7169s^2+1.0255s+0.3791}$
5	$\frac{0.1789}{s^5+1.1725s^4+1.9374s^3+1.3096s^2+0.7525s+0.1789}$
6	$\frac{0.0895}{s^6+1.1592s^5+2.1718s^4+1.5898s^3+1.1719s^2+0.4324s+0.0948}$

Chebyshev lowpass prototype transfer functions with 1 dB ripple ($\varepsilon = 0.5088$)

n	$H_P(s)$
1	$\frac{1.9652}{s+1.9652}$
2	$\frac{0.9826}{s^2+1.0977s+1.1025}$
3	$\frac{0.4913}{s^3+0.9883s^2+1.2384s+0.4913}$
4	$\frac{0.2456}{s^4+0.9528s^3+1.4539s^2+0.7426s+0.2756}$
5	$\frac{0.1228}{s^5+0.9368s^4+1.6888s^3+0.9744s^2+0.5805s+0.1228}$
6	$\frac{0.0614}{s^6+0.9283s^5+1.9308s^4+1.20121s^3+0.9393s^2+0.3071s+0.0689}$

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Normalized Chebyshev Function

- The Chebyshev magnitude response function with an order of n and the normalized cutoff frequency $\omega = 1$ radian per second is given by

$$|B_n(\omega)| = \frac{1}{\sqrt{1+\varepsilon^2 C_n{}^2(\omega)}}, \; n \geq 1$$

- where the function $C_n(\omega)$ is defined as

$$C_n(\omega) = \begin{cases} \cos\left(n\cos^{-1}(\omega)\right) & \omega \leq 1 \\ \cosh\left(n\cosh^{-1}(\omega)\right) & \omega > 1 \end{cases}$$

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Normalized Chebyshev Function

- ε is the ripple specification on the filter passband.
  - $cosh^{-1}(x) = ln(x + \sqrt{x^2 - 1})$
- a factor from the real pole [s + sinh(β)], it follows that
  - $B_n(s) = \frac{K}{[s+sinh(\beta)] \prod_{k=0}^{n-1/2-1}(s^2+b_k s+c_k)}$
  - $a_k = \frac{(2k+1)\pi}{2n}$ $for\ k = 0, 1, \ldots \ldots, {}^{(n-1)}/_{2-1}$
  - b_k=2sin (α_k)sinh(β)
  - $c_k = [sin\,(\alpha_k) sin\,(\beta)]^2 + [cos(\alpha_k) cosh(\beta)]^2$

Normalized Chebyshev Function

- For the unit passband gain and the filter order as an odd number, we set : $\boldsymbol{B_n(0)}$=1
  - K= sinh (β) $\prod_{K=0}^{n-1/2-1} c_k$
  - β= $sinh^{-1}\left(\frac{1/\varepsilon}{n}\right)$
  - $sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$
- Following a similar procedure for the even number of n, we have
  - $B_n(s) = \frac{k}{\prod_{k=0}^{n/2-1} s^2 + b_k s + c_k}$
  - α_k=(2k+1)π/(2n) for k = 0,1,……,n/2 – 1
  - b_k=2sin (α_k)sinh(β)
  - $c_k = [sin\,(\alpha_k sinh\,(\beta))]^2 + [\cos(\alpha_k cosh(\beta))]^2$

Normalized Chebyshev Function

- For the unit passband gain and the filter order as an even
- number, we require that $B_n(0)={}^{1}/_{\sqrt{1+\varepsilon^2}}$ so that the maximum
- magnitude of the ripple on passband equals 1.

- $\mathrm{K} = \prod_{k=0}^{n/2-1} {}^{c_k}/_{\sqrt{1+\varepsilon^2}}$
- $\mathrm{B}=\boldsymbol{sinh^{-1}}({}^{1/\varepsilon}/_{n})$

$$\boldsymbol{sinh^{-1}(x) = ln(x + \sqrt{x^2 + 1})}$$

Conversion from analog filter specifications to low pass prototype specifications(table 4)

- The normalized stopband frequency vs can be determined from the frequency specifications of an analog filter in Table

Analog Filter Specifications	Lowpass Prototype Specifications
Lowpass: ω_{ap}, ω_{as}	$v_p = 1$, $v_s = \omega_{as}/\omega_{ap}$
Highpass: ω_{ap}, ω_{as}	$v_p = 1$, $v_s = \omega_{ap}/\omega_{as}$
Bandpass: ω_{apl}, ω_{aph}, ω_{asl}, ω_{ash} $\omega_0 = \sqrt{\omega_{apl}\omega_{aph}}$, $\omega_0 = \sqrt{\omega_{asl}\omega_{ash}}$	$v_p = 1, v_s = \frac{\omega_{ash} - \omega_{asl}}{\omega_{aph} - \omega_{apl}}$
Bandstop: ω_{apl}, ω_{aph}, ω_{asl}, ω_{ash} $\omega_0 = \sqrt{\omega_{apl}\omega_{aph}}$, $\omega_0 = \sqrt{\omega_{asl}\omega_{ash}}$	$v_p = 1, v_s = \frac{\omega_{aph} - \omega_{apl}}{\omega_{ash} - \omega_{asl}}$

ω_{ap}, passband frequency edge; ω_{as}, stopband frequency edge; ω_{apl}, lower cutoff frequency in passband; ω_{aph}, upper cutoff frequency in passband; ω_{asl}, lower cutoff frequency in stopband; ω_{ash}, upper cutoff frequency in stopband; ω_o, geometric center frequency.

Lowpass and Highpass Filter Design Examples

a. Design a digital lowpass Butterworth filter with the following specifications:

1. 3 dB attenuation at the passband frequency of 1.5 kHz
2. 10 dB stopband attenuation at the frequency of 3 kHz

Solution : 1.First ,we obtain the digital frequencies in radians per second:

$$\omega_{dp}=2\pi f= 2\pi(1500)=3000\pi \text{ rad/sec}$$

$$\omega_{ds}= 2\pi f= 2\pi(3000)=6000\pi \text{ rad/sec}$$

- $$T=1/f_s = \frac{1}{8000} \text{ sec}$$

We apply the warping equation as

$$\omega_{ap} = \frac{2}{T} tan \frac{\omega_d T}{2}=16000\times tan\left(\frac{3000\pi/8000}{2}\right)=1{,}0691\times 10^4 \text{rad/sec}$$

$$\omega_{as} = \frac{2}{T} tan \frac{\omega_d T}{2}=16000\times tan\left(\frac{6000\pi/8000}{2}\right)=3{,}8627\times 10^4 \text{rad/sec}$$

We then find the lowpass prototype specifications usinig the Table 4.

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Lowpass and Highpass Filter Design Examples

$$v_s = \omega_{as}/\omega_{ap} = 3{,}862\times \frac{10^4}{1{,}0691\times 10^4}$$

$$= 3{,}6130 \text{ rad/sec and } A_s = 10\text{dB}$$

- The first order is computed as

$$\varepsilon^2 = 1\,, n = 0{,}8553$$

2 . Rounding n up ,we choose n=1 for the lowpass prototype .From table 3 we have $H_p(s) = \frac{1}{s+1}$

- yields the analog filter :

$$\text{H(s)} = H_p(s)\Big|_{\frac{s}{\omega ap}} = \frac{1}{\frac{s}{\omega ap}+1} = \frac{\omega_{ap}}{s+\omega_{ap}} = \frac{1{,}0691\times 10^4}{s+1{,}0691\times 10^4}$$

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Bandpass and Bandstop Filter Design Examples

- *Design a second-order digital bandpass Butterworth filter with the following specifications:*
  - *an upper cutoff frequency of 2.6 kHz and*
  - *a lower cutoff frequency of 2.4 kHz,*
  - *a sampling frequency of 8,000 Hz*

- Let us find the digital frequencies in radians per second:
- $\omega_h\ 2\pi f_h = 2\pi(2600) = 5200\pi$ rad/sec
- $\omega_l 2\pi\ \ f_l = 2\pi(2400) = 4800\pi$ rad/sec, and T = 1/ fs =1/8000 sec.

Bandpass and Bandstop Filter Design Examples

Following the steps of the design procedure, we have the following:

$\omega_{ah} = \frac{2}{T}\tan\frac{\omega_h \mathrm{T}}{2}$ =16000× $\tan\frac{5200\pi/8000}{2} = 2{,}6110 \times\ 10^4 rad/sec$

ω_{al} =16000× $\tan\frac{\omega_l T}{2} = 16000 \times \tan(0{,}3\pi) = 2{,}2022 \times 10^4 rad/sec$

W=ω_{ah} - ω_{al} *=26110-22022=4088 rad/sec*

$\omega_0{}^2 = \omega_{ah} \times \omega_{al} = 5{,}7499 \times 10^8$

lowpass prototype with the order of 1 to produce the bandpass filter with the order of 2, as $H_P(s) = \frac{1}{s+1}$

the lowpass-to-bandpass transformation, it follows that

H(s)= $\frac{W_s}{s^2+W_s+\omega_0{}^2} = \frac{4088s}{s^2+4088s+5{,}7499\times10^8}$

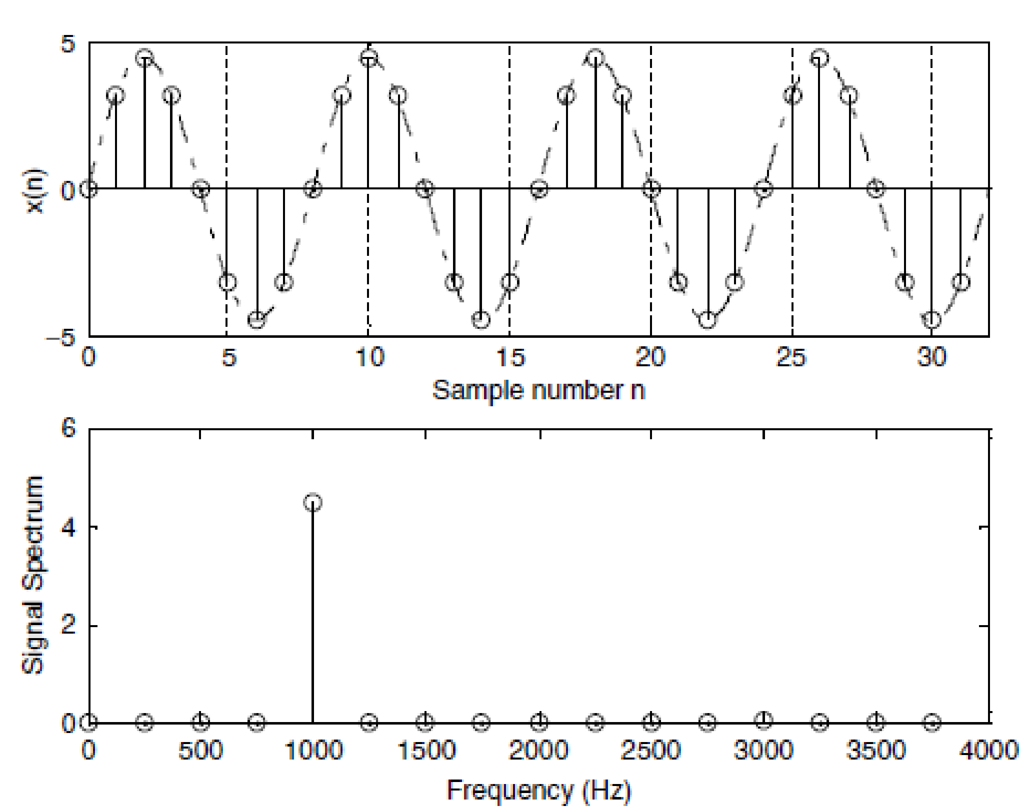
Hence we apply the BLT to yield

H(z)= $\left.\frac{W_s}{s^2+4088s+5{,}7499\times10^8}\right|_{s=16000(z-1)/(z+1)}$

Via algebra work, we obtain the digital filter as : *H(z)=*$\frac{0{,}0730-0{,}0730^{-2}}{1+0{,}7117z^{-1}+0{,}8541z^{-2}}$

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Discrete Fourier Transform



Sampled signal x(n) obtained by sampling the continuous signal x(t) at a sampling rate of f_s=8kHz.

In time domain, representation of digital signals describes the signal amplitude versus the sampling time instant or the sample number.

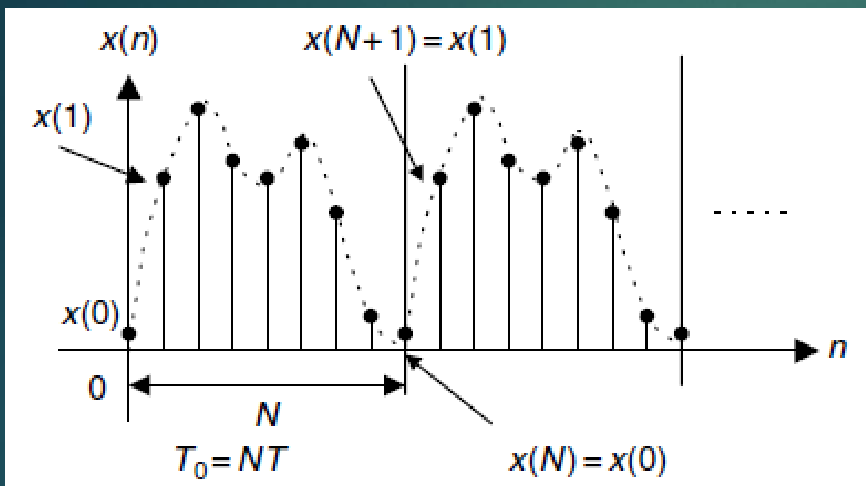
In a vast number of applications, signal frequency content is very useful.

The representation of the digital signal in terms of its frequency components in the frequency domain, the **signal spectrum**, needs to be developed.

The algorithm transforming the time domain signal samples to the frequency domain components is known as the **Discrete Fourier Transform**, (**DFT**).

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Periodic Digital Signals



It is supposed that we estimate the spectrum of a periodic digital signal ***x(n)*** resulted from the signal ***x(t)*** sampled at a rate of f_s Hz ($T = {}^{1}/_{f_s}$ is the sampling period) with fundamental period T_0:

$$T_0 = NT$$

where there are N samples within the duration of the fundamental period.

The periodic digital signal is assumed to be band limited to have all harmonic frequencies less than the folding frequency $f_s/2$.

According to Fourier series, the periodic signal ***x(t)*** in exponential complex form is:

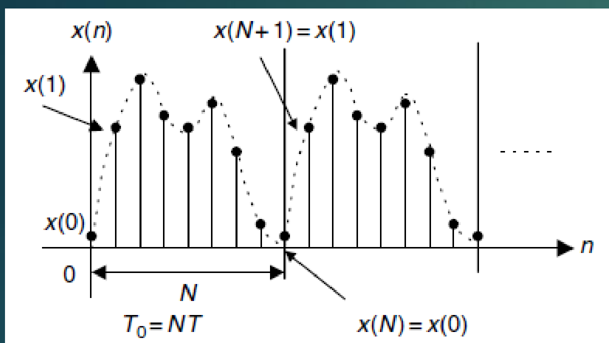
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$c_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt \quad -\infty < k < \infty$$

where k is the number of harmonics corresponding to the harmonic frequency of kf_0, where $f_0 = {}^{1}/_{T_0}$ is the fundamental frequency.

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Fourier coefficients of Periodic Digital Signals



Substituting $\omega_0 = 2\pi/T_0$, $T_0 = NT$, $t = nT$, $dt = T$ we obtain:

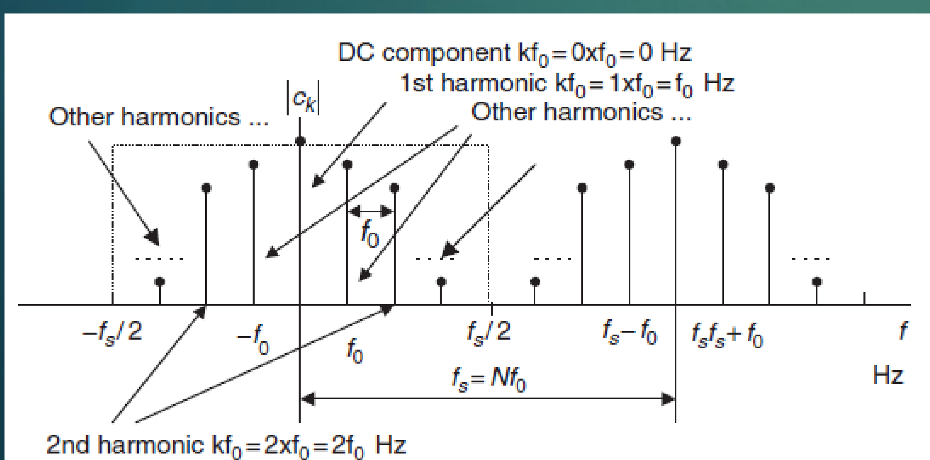
$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}} \qquad -\infty < k < \infty$$

The resultant spectrum of complex c_k will be two-sided. Very important feature is:

$$c_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi(k+N)n}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}\ e^{-j2\pi n}$$

Due to the fact that: $e^{-j2\pi n} = cos(2\pi n) - jsin(2\pi n) = 1$

$$c_{k+N} = c_k$$



Thus, we may compute the spectrum over the range from 0 to f_s Hz with nonnegative indices:

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}} \quad k = 0, 1, \ldots N-1$$

For k harmonic the corresponding frequency is kf_0, and f_0 is the **frequency resolution**.

Anna Daskalaki | d_anna-13-ntora@windowslive.com

Fourier coefficients of Periodic Digital Signals example

Given a sequence x(n) for $0 \leq n \leq 3$,whwre x(0)= 1,x(1)= 2, x(2)=3 and x(3)=4,

a. Evaluate its DFT X(k)

Solution :

Since N=4 and $W_4 = e^{-j\frac{\pi}{2}}$

$$X(k)=\sum_{n=0}^{3} x(n)W_4^{kn}=\sum_{n=0}^{3} x(n)e^{-j\frac{\pi kn}{2}}$$

For k=0

$X(0)=\sum_{n=0}^{3} x(n)e^{-j0} = x(0)e^{-j0} + x(1)e^{-j0}+x(2)e^{-j0} +x(3)e^{-j0}=$

=x(0)+ x(1)+ x(2)+x(3)

=1+2+3+4=10

For k=1

$X(1)=\sum_{n=0}^{3} x(n)e^{-j\frac{\pi n}{2}} = x(0)e^{-j0} + x(1)e^{-j\frac{\pi}{2}}+x(2)e^{-j\pi} +x(3)e^{-j\frac{3\pi}{2}}=$

=x(0)-j x(1)+ x(2)+j x(3)

=1 – j2 -3 +j4 =-2 +j2

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Fourier coefficients of Periodic Digital Signals example

For k=2

X(2)=$\sum_{n=0}^{3} x(n)e^{-j\pi n} = x(0)e^{-j0} + x(1)e^{-j\pi}+x(2)e^{-j2\pi} +x(3)e^{-j3\pi}=$

=x(0)- x(1)+ x(2)-x(3)

=1-2+3-4=-2

For k=3

X(3)=$\sum_{n=0}^{3} x(n)e^{-j\frac{3\pi n}{2}} = x(0)e^{-j0} + x(1)e^{-j\frac{3\pi n}{2}}+x(2)e^{-j3\pi} +x(3)e^{-j\frac{9\pi}{2}}=$

=x(0)+ξ x(1)- x(2)-jx(3)

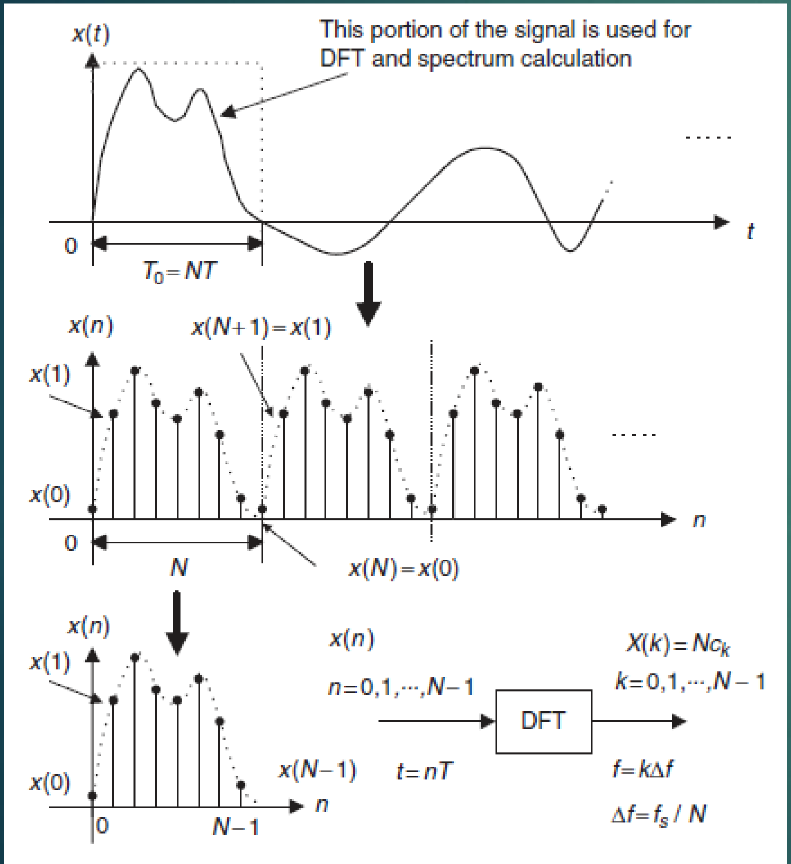
=1+j2-3-j4 = -2-j2

Let us verify the result using the MATLAB function fft():

- X = fft([1 2 3 4])
- X = 10.0000 2.0000 + 2.0000i -2.0000 2.0000 -2.0000i

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Discrete Fourier Transform formulas



__Procedure__: The process acquires data samples by digitizing the interested continuous signal ***x(t)*** for a duration of $\boldsymbol{T_0 = NT}$ seconds.

It is assumed that a periodic signal ***x(n)*** is obtained by copying the acquired N data samples, with duration $\boldsymbol{T_0}$, to itself repetitively.

Further, it is assumed continuity between the N data sample frames (__not__ true in practice…).

From Fourier coefficients, using one-period N data samples, we compute the **DFT coefficients**:

$$\boldsymbol{X(k) = Nc_k = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}} \quad \boldsymbol{k} = 0, 1, \dots N-1$$

DFT with N data samples of ***x(n)***, at a sampling rate of f_s Hz (T= $^1/_{f_s}$), produces N complex DFT coefficients $\boldsymbol{X(k)}$. For $\boldsymbol{k}$ harmonic the corresponding frequency is $\boldsymbol{k\frac{f_s}{N}}$ and $\boldsymbol{\frac{f_s}{N}}$ is the **frequency resolution**.

Anna Daskalaki | d_anna-13-ntora@windowslive.com

Discrete Fourier Transform (DFT) definition

Given a sequence x(n), $0 \leq n \leq N-1$ its **DFT** is defined as:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad k = 0, 1, \dots N-1$$

the factor W_N (termed also twiddle factor) is defined as: $W_N = e^{-j\frac{2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right)$

The **inverse DFT** is given by:

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi kn}{N}} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \quad n = 0, 1, \dots N-1$$

In time domain we use the sample number or time index n for indexing the digital sample sequence x(n). However, in frequency domain, we use index k for indexing N calculated DFT coefficients X(k). We also refer to k as the frequency bin number. The frequency bin k can be mapped to its corresponding frequency:

$$\omega = \frac{k\omega_s}{N} \Leftrightarrow f = \frac{kf_s}{N}$$

Similarly, frequency resolution is defined:

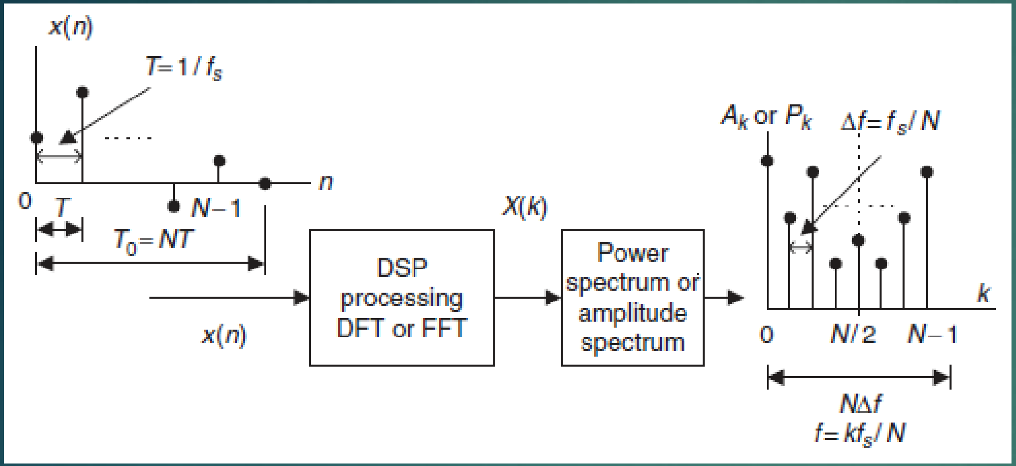
$$\Delta\omega = \frac{\omega_s}{N} \Leftrightarrow \Delta f = \frac{f_s}{N}$$

MATLAB FFT functions.

X = fft(x)	% Calculate DFT coefficients
x = ifft(X)	% Inverse DFT
x = input vector	
X = DFT coefficient vector	

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Amplitude Spectrum and Power Spectrum



By applying the DFT to the truncated sequence x(n), with range $0 \leq n \leq N-1$, we get the N **DFT coefficients**:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad k = 0, 1, \ldots N-1$$

Since each calculated DFT coefficient is a complex number, the magnitude and phase of each DFT coefficient can be determined and plotted versus its frequency index (we refer to them as the amplitude spectrum and phase spectrum, respectively). We define the **amplitude spectrum** as:

$$A_k = \frac{1}{N}|X(k)| = \frac{1}{N}\sqrt{(Real[X(k)])^2 + (Imag[X(k)])^2} \quad k = 0, 1, \ldots N-1$$

The amplitude spectrum can be modified to a one-sided amplitude spectrum by doubling the amplitudes, but keeping the original DC term:

$$\widehat{A_k} = \begin{cases} \frac{1}{N}|X(0)|, & k = 0 \\ \frac{2}{N}|X(k)|, & k = 1, \ldots {}^{N}/_{2} \end{cases}$$

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Amplitude Spectrum and Power Spectrum

Correspondingly, the **phase spectrum** is given by: $\varphi_k = tan^{-1}\left(\frac{Imag[X(k)]}{Real[X(k)]}\right) \quad \boldsymbol{k} = 0, 1, \dots N-1$

Besides the amplitude spectrum, the power spectrum is also used. The DFT power spectrum is defined as:

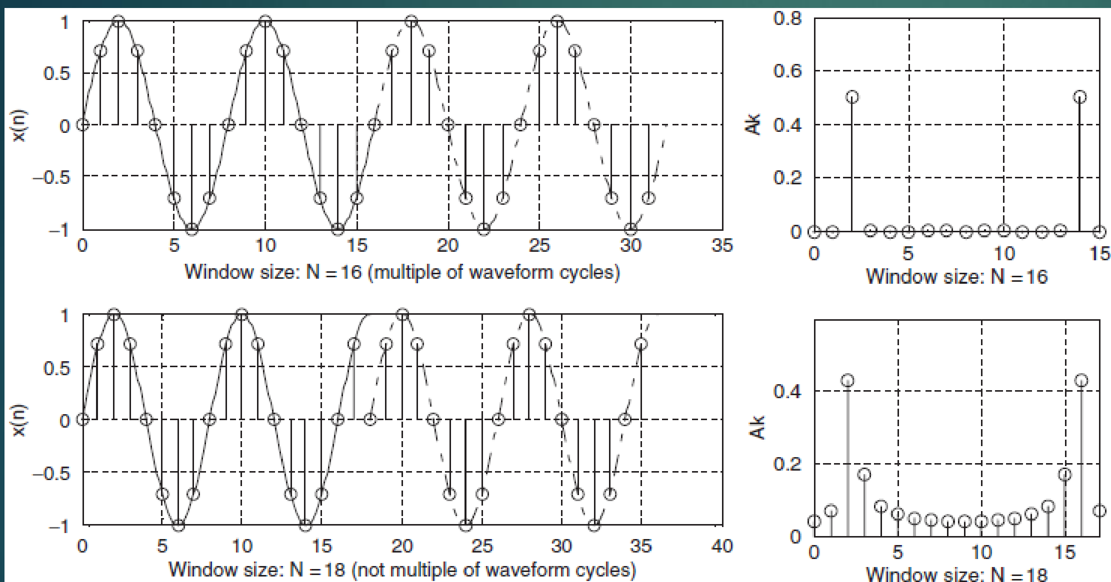
$$\boldsymbol{P_k} = \frac{\mathbf{1}}{\boldsymbol{N^2}}|\boldsymbol{X}(\boldsymbol{k})|^2 = \frac{\mathbf{1}}{\boldsymbol{N^2}}\left\{(\boldsymbol{Real}[\boldsymbol{X}(\boldsymbol{k})])^2 + (\boldsymbol{Imag}[\boldsymbol{X}(\boldsymbol{k})])^2\right\} \quad \boldsymbol{k} = 0, 1, \dots N-1$$

The power spectrum can be also modified to a one-sided amplitude spectrum by doubling the amplitudes, but keeping the original DC term:

$$\widehat{P_k} = \begin{cases} \frac{1}{\boldsymbol{N^2}}|\boldsymbol{X}(\boldsymbol{k})|^2, & k = 0 \\ \frac{2}{\boldsymbol{N^2}}|\boldsymbol{X}(\boldsymbol{k})|^2, & k = 1, \dots {}^{N}/_{2} \end{cases}$$

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Spectral Estimation Using Window Functions



When apply DFT to sampled data, we theoretically imply the following assumptions:

- the sampled data are periodic to themselves, and
- the sampled data are continuous to themselves and band limited to the folding frequency.

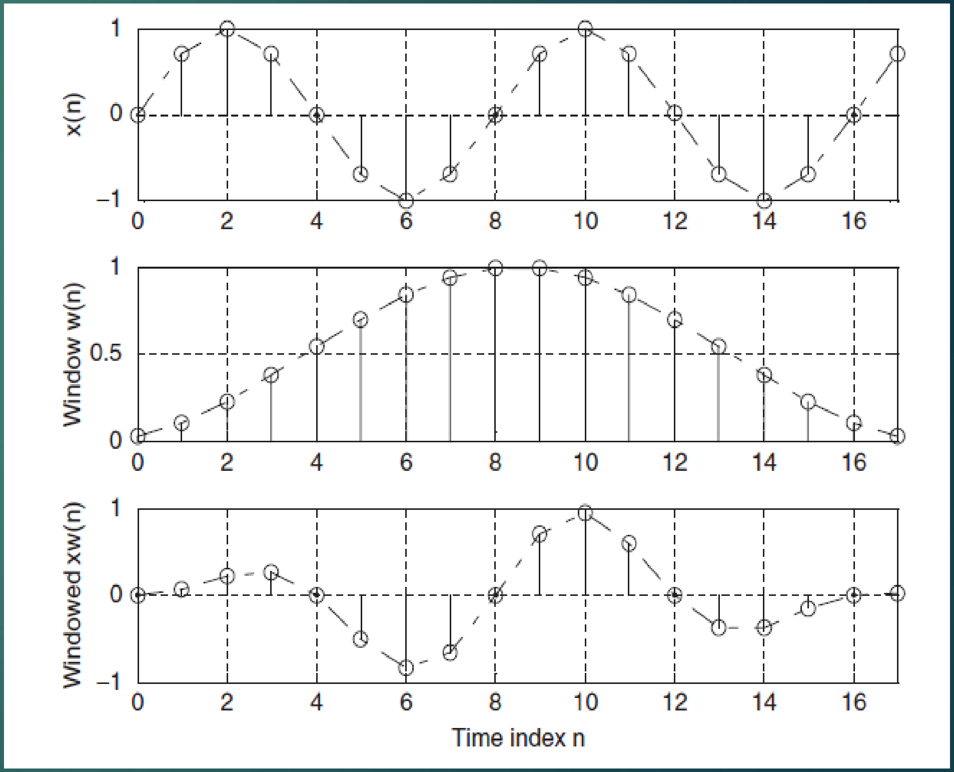
The last assumption is often violated, thus the discontinuity produces undesired harmonic frequencies. This effect is termed **spectral leakage**. The amount of spectral leakage is close related to amplitude discontinuity in time domain. The bigger the discontinuity, the more the leakage.

Spectral Estimation Using Window Functions example

In Figure given

- x(2) = 1and w(2)= 0,2265
- x(5)= - 0,7071 and w(5) = 0,7008

a. Calculate the windowed sequence data point $x_w\ (2) and\ x_w(5)$.



Spectral Estimation Using Window Functions example

Applying the window function operation leads to

$x_w(2)$=x(2)× $w(2) = 1 \times 0{,}2265 = 0{,}2265$

$x_w(5)$=x(5)× $w(5) = 0{,}701 \times 0{,}7008 = -0{,}4956$

The common windows functions are listed as follows. The common window(no window function): $w_R(n) = 1 \quad 0 \le n \le N-1$

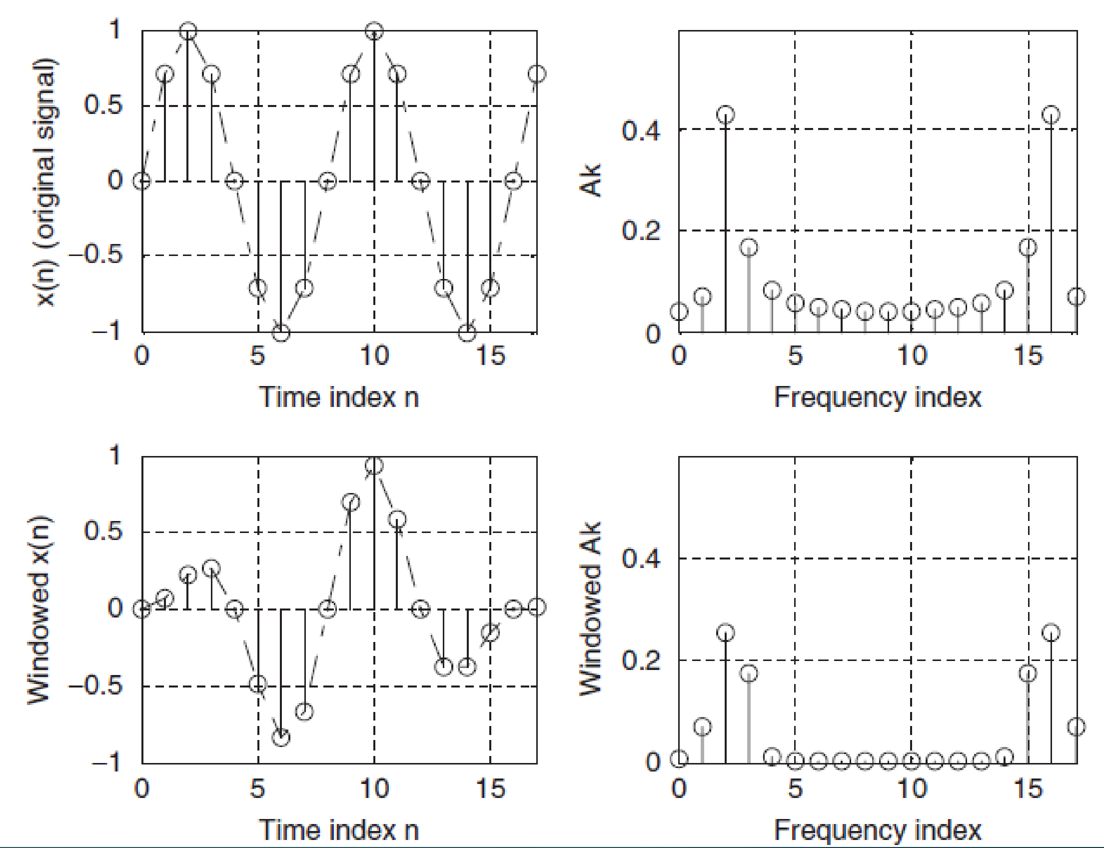
The triangular window

$$w_{tri}(n) = 1 - \frac{|2n-N-1|}{N-1}, 0 \le n \le N-1$$

The Hamming window :

$$(n) = 0{,}54 \; - 0{,}46 \cos\left(\frac{2\pi n}{N-1}\right)$$

$,0 \le n \le$ N-1



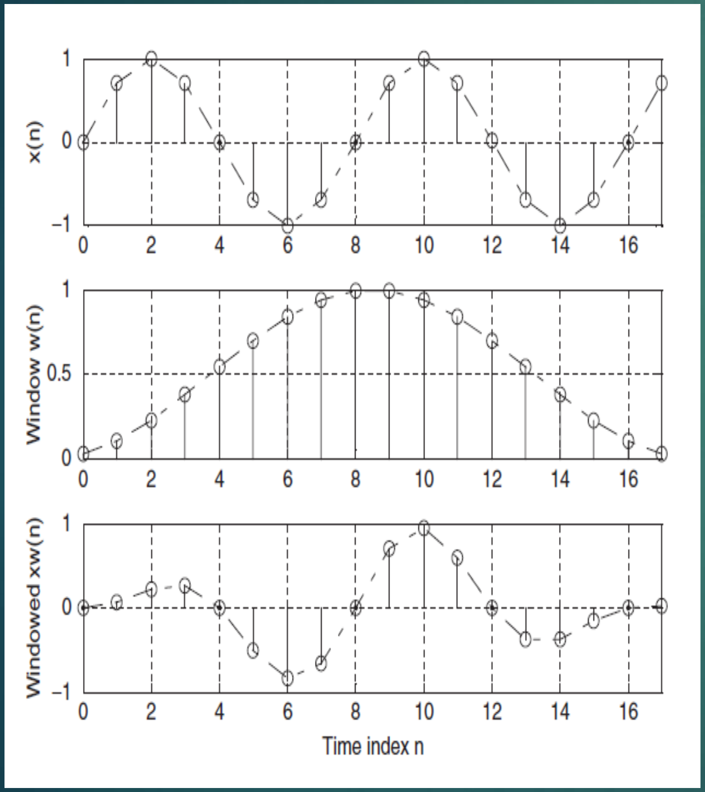
Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

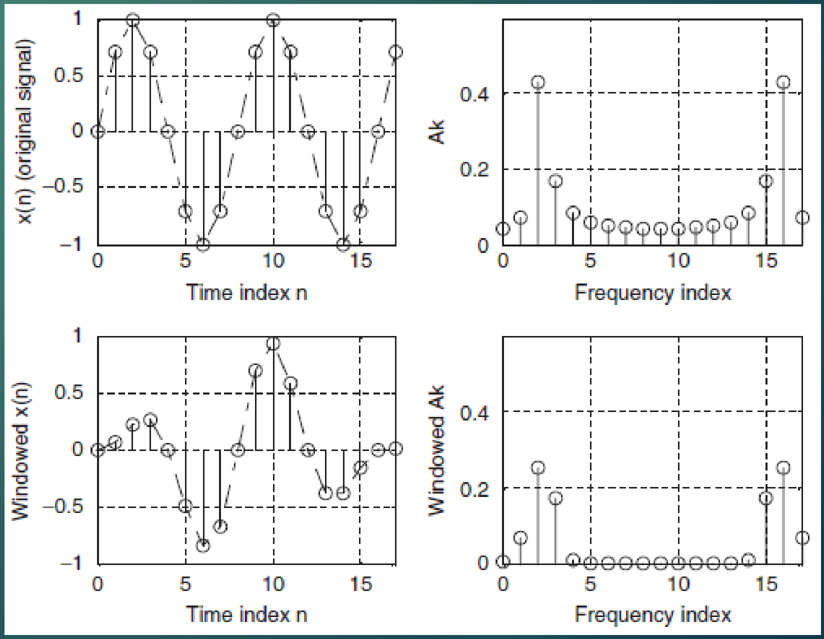
Spectral Estimation Using Window Functions

To reduce the effect of spectral leakage, a window function can be used whose amplitude tapers smoothly and gradually toward zero at both ends.

We apply the window function w(n) to a data sequence x(n) to obtain a *windowed* sequence, $\boldsymbol{x_w(n)}$:

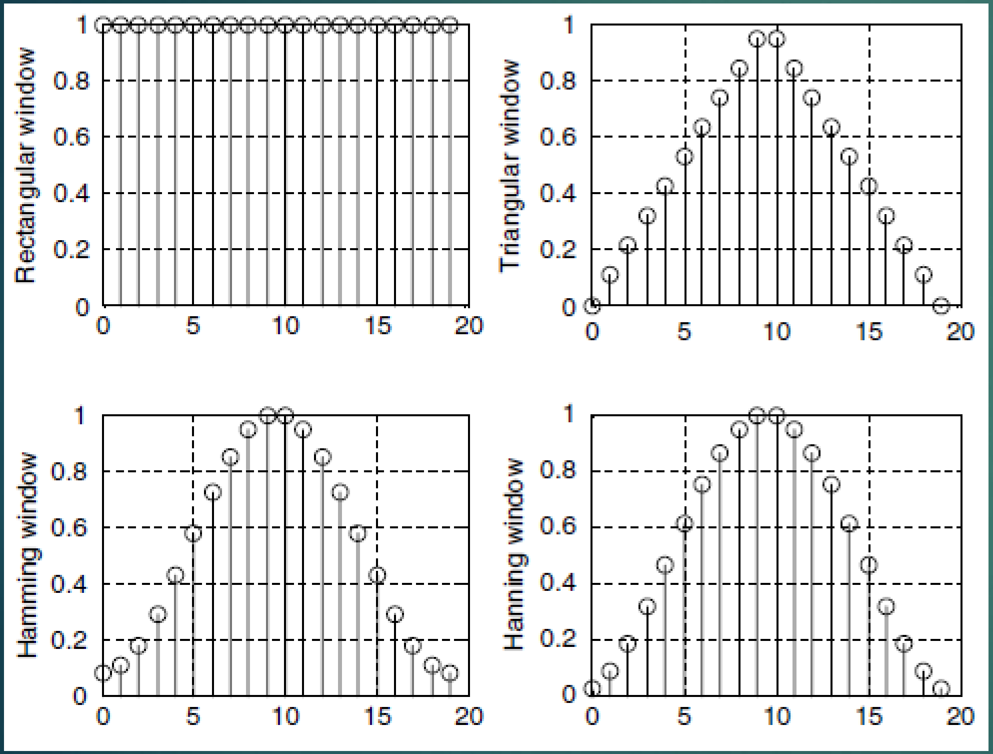
$$\boldsymbol{x_w(n) = x(n) \cdot w(n)} \qquad \boldsymbol{n} = 0, 1, \ldots N-1$$





Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Spectral Estimation Using Window Functions



Rectangular window (no window function):

$$w_{rec}(n) = 1 \qquad 0 \le n \le N-1$$

Triangular window:

$$w_{tri}(n) = 1 - \frac{|2n-N+1|}{N-1} \qquad 0 \le n \le N-1$$

Hamming window:

$$w_{ham}(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)$$

$$0 \le n \le N-1$$

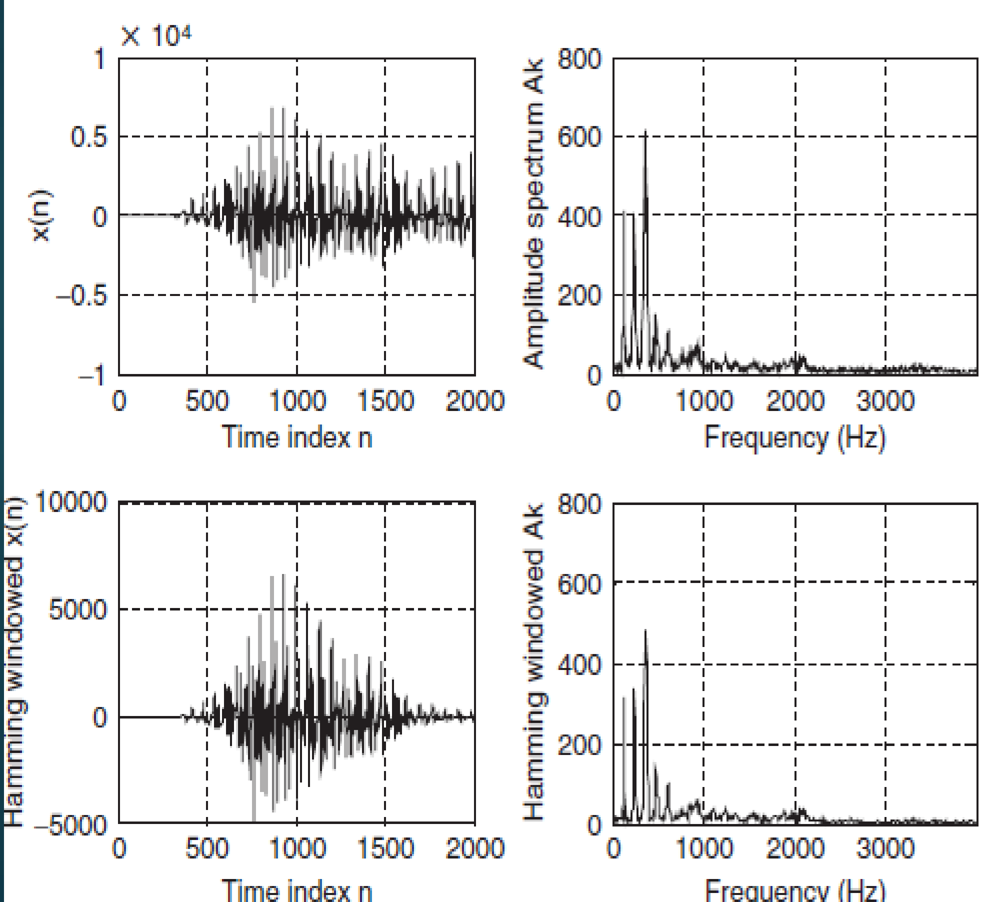
Hanning window:

$$w_{han}(n) = 0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right)$$

$$0 \le n \le N-1$$

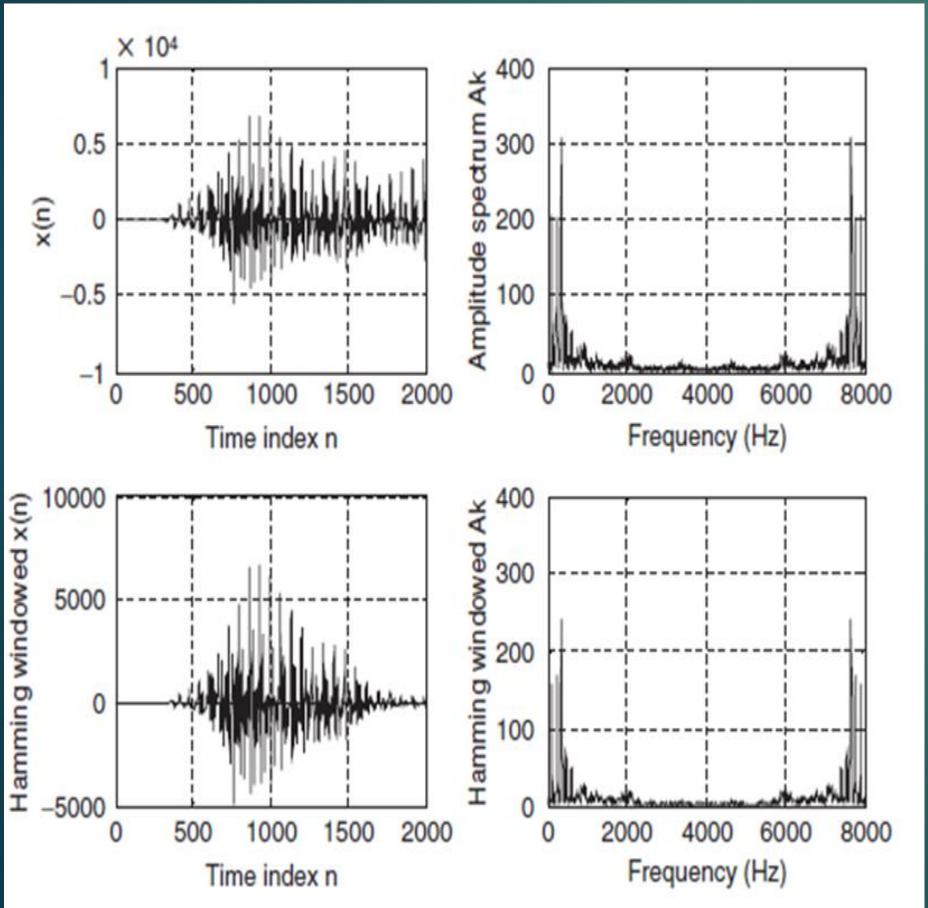
Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Application to Speech Spectral Estimation



- the comparisons of amplitude spectral estimation for speech data (we.dat) with 2,001 samples and a sampling rate of 8,000 Hz using the rectangular window (no window) function and the Hamming window function.
- one-sided spectrum
- when data length is short reduction of spectral leakage using a window function will come to be prominent.

Application to Speech Spectral Estimation



- the comparisons of amplitude spectral estimation for speech data (we.dat) with 2,001 samples and a sampling rate of 8,000 Hz using the rectangular window (no window) function and the Hamming window function.
- two-sided spectrum
- the data length of the sequence increases, the frequency resolution will be improved and spectral leakage will become less significant.

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

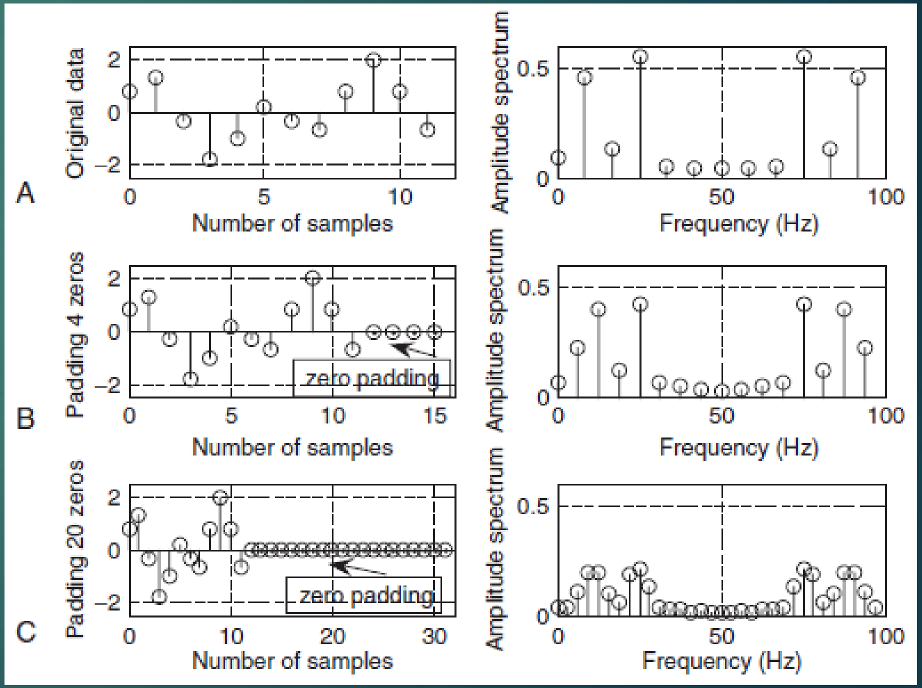
Fast Fourier Transform (FFT)

Fast Fourier Transform (FFT) is a very efficient algorithm in computing DFT coefficients and can reduce a very large amount of computational complexity (multiplications). For a data length of N:

- Complex multiplications of **DFT**: N^2
- Complex multiplications **of FFT**: $\frac{N}{2} log_2(N)$

Prerequisite of FFT is that the digital sequence ***x(n)*** consists of 2^m samples, where m is a positive integer. If ***x(n)*** contain $N \neq 2^m$ samples, then we simply append it with zeros (zero padding) until the number of the appended sequence is $\overline{N} = 2^m$ samples:

$$\overline{x}(n) = \begin{cases} x(n) & 0 \leq n \leq N-1 \\ 0 & N \leq n \leq \overline{N}-1 \end{cases}$$



We focus on two FFT formats that are referred to as the **radix-2 FFT algorithms**:

- Decimation-in-frequency algorithm
- Decimation-in-time algorithm

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Method of Decimation-in-Frequency

Recall the DFT definition, provided that $N = 2^m$:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad k = 0, 1, \dots N-1$$

The above equation is split into:

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n)\, W_N^{kn}$$

Modifying the second term:

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{kn} + W_N^{\frac{N}{2}k} \sum_{n=0}^{\frac{N}{2}-1} x(n+\frac{N}{2})\, W_N^{kn}$$

and due to the fact that $W_N^{\frac{N}{2}} = -1$, it results:

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^k x(n+\frac{N}{2})\right] W_N^{kn}$$

DFT algorithm is now split to even and odd frequency bins, $k = 2q$ and $k = 2q+1$:

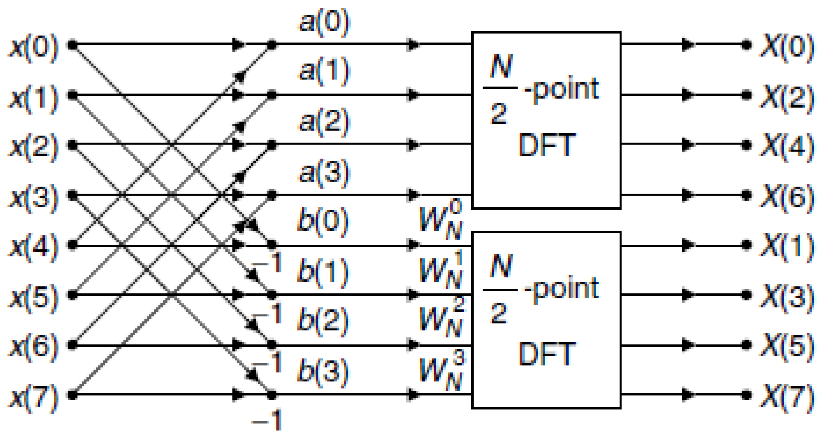
$$X(2q) = \sum_{n=0}^{\frac{N}{2}-1} \alpha(n)\, W_{N/2}^{qn} = \text{DFT}\{\alpha(n) \text{ with } \tfrac{N}{2} \text{ points}\},\ \alpha(n) = x(n) + x\left(n+\frac{N}{2}\right)$$

$$X(2q+1) = \sum_{n=0}^{\frac{N}{2}-1} b(n) W_N^n\, W_{N/2}^{qn} = \text{DFT}\{b(n) W_N^n \text{ with } \tfrac{N}{2} \text{ points}\},\ b(n) = x(n) - x\left(n+\frac{N}{2}\right)$$

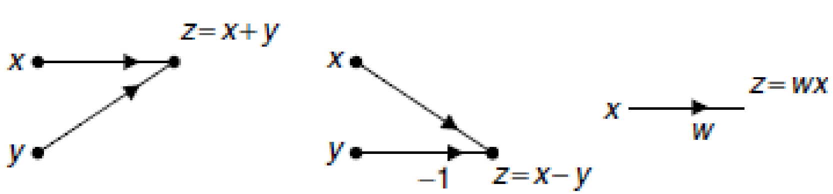
using that: $W_N^2 = e^{-j\frac{2\pi\times 2}{N}} = e^{-j\frac{2\pi}{N/2}} = W_{N/2}$

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Method of Decimation-in-Frequency



The first iteration of the eight-point FFT.



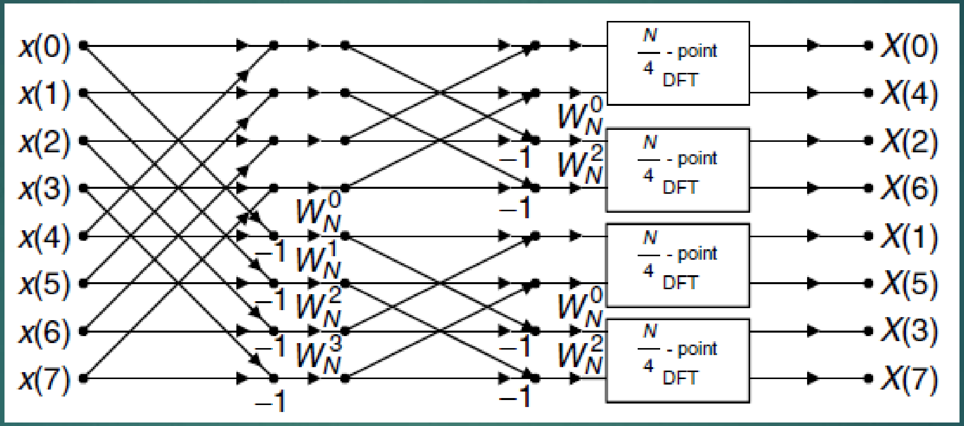
Definitions of the graphical operations.

Input Data	Index Bits	Reversal Bits	Output Data
$x(0)$	000	000	$X(0)$
$x(1)$	001	100	$X(4)$
$x(2)$	010	010	$X(2)$
$x(3)$	011	110	$X(6)$
$x(4)$	100	001	$X(1)$
$x(5)$	101	101	$X(5)$
$x(6)$	110	011	$X(3)$
$x(7)$	111	111	$X(7)$

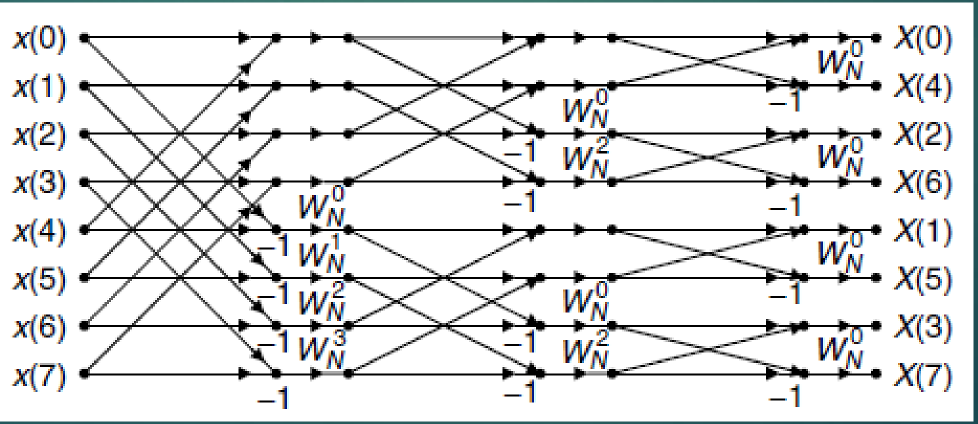
Binary	index	1st split	2nd split	3rd split	Bit reversal
000	0	0	0	0	000
001	1	2	4	4	100
010	2	4	2	2	010
011	3	6	6	6	110
100	4	1	1	1	001
101	5	3	5	5	101
110	6	5	3	3	011
111	7	7	7	7	111

Bit reversal process in FFT

Method of Decimation-in-Frequency



The first and second iteration of eight-point FFT



The eight-point FFT (3 iterations, 12 complex multiplications)

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

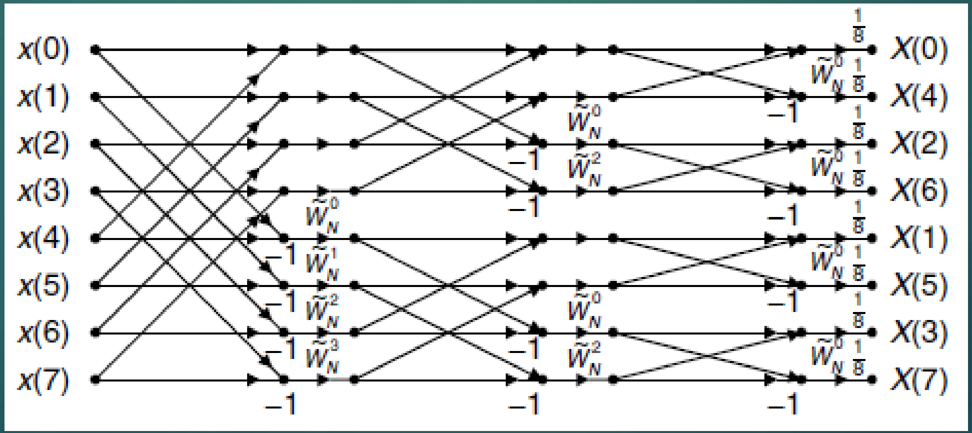
Method of Decimation-in-Frequency

Compare **DFT** and **inverse DFT** definitions:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad k = 0, 1, \dots N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, \widetilde{W}_N^{kn} \quad n = 0, 1, \dots N-1$$

The main differences are that, for inverse FFT, the twiddle factor W_N is changed to be $\widetilde{W}_N = W_N^{-1}$, and the sum is multiplied by a factor of $\frac{1}{N}$. Thus, by modifying accordingly the previous FFT block diagram, the inverse FFT block diagram is achieved:



The eight-point inverse FFT

Anna Daskalaki | d_anna-13-ntora@windowslive.com

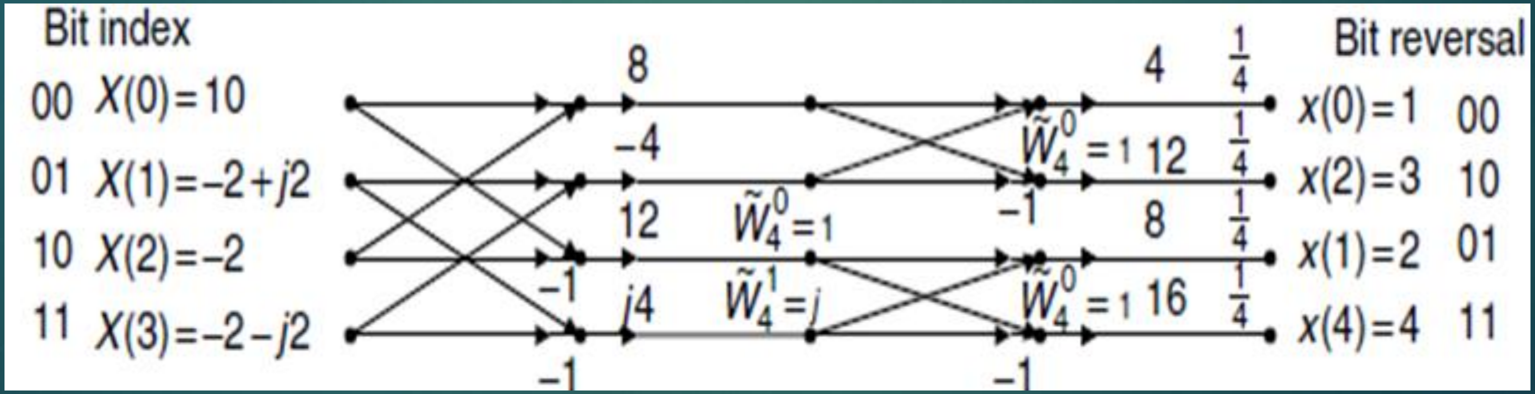
Method of Decimation-in-Frequency Example

Given the DFT sequence $x_{(k)}$ for $0 \leq k \leq 3$

a. Evaluate its inverse DFT $x_{(n)}$ using the decimation in frequency FFT method.

<u>Solution :</u>

a. Using the inverse FFT blocks diagram , we have



Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Method of Decimation-in-Time

The input sequence $\boldsymbol{x(n)}$ is split into the even indexed $\boldsymbol{x(2q)}$ and the odd indexed $\boldsymbol{x(2q+1)}$ sequences, each with $\frac{N}{2}$ data points. The DFT definition becomes (using $\boldsymbol{W_N^2}=\boldsymbol{W}_{N/2}$):

$$X(\boldsymbol{k}) = \sum_{q=0}^{\frac{N}{2}-1} x(2q)W_{N/2}^{qk} + W_N^k \sum_{q=0}^{\frac{N}{2}-1} x(2q+1)W_{N/2}^{qk} \qquad \boldsymbol{k} = 0, 1, \ldots N-1$$

DFT algorithm is now expressed as follows:

$$G(\boldsymbol{k}) = \sum_{q=0}^{\frac{N}{2}-1} x(2q)W_{N/2}^{qk} = \text{DFT}\left\{x(2q) \text{ with } \frac{N}{2} \text{ points}\right\}, \; G(\boldsymbol{k}) = G\left(k+\frac{N}{2}\right) \quad \boldsymbol{k} = 0, 1, \ldots \frac{N}{2}-1$$

$$H(\boldsymbol{k}) = \sum_{q=0}^{\frac{N}{2}-1} x(2q+1)W_{N/2}^{qk} = \text{DFT}\left\{x(2q) \text{ with } \frac{N}{2} \text{ points}\right\}, \; H(\boldsymbol{k}) = H\left(k+\frac{N}{2}\right) \quad \boldsymbol{k} = 0, 1, \ldots \frac{N}{2}-1$$

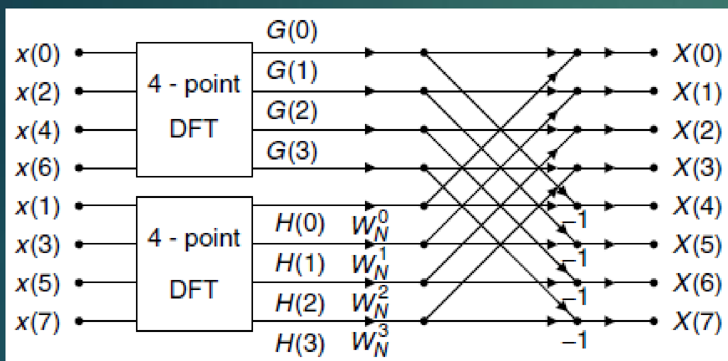
$$\mathrm{X}(\boldsymbol{k}) = G(\boldsymbol{k}) + W_N^k H(\boldsymbol{k}) \qquad \boldsymbol{k} = 0, 1, \ldots \frac{N}{2}-1$$

$$\mathrm{X}\left(\frac{N}{2}+\boldsymbol{k}\right) = G(\boldsymbol{k}) - W_N^k H(\boldsymbol{k}) \qquad \boldsymbol{k} = 0, 1, \ldots \frac{N}{2}-1$$

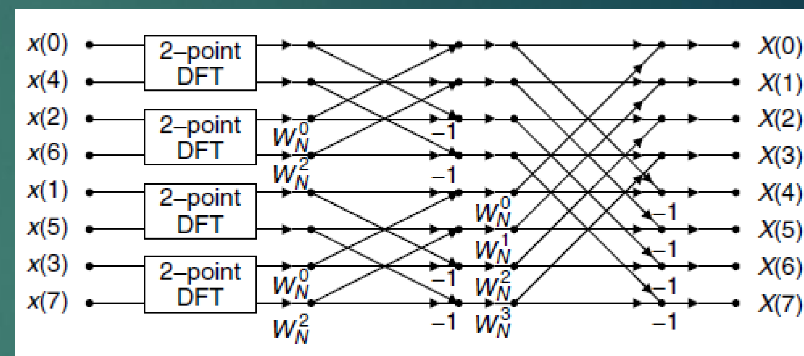
Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Method of Decimation-in-Time

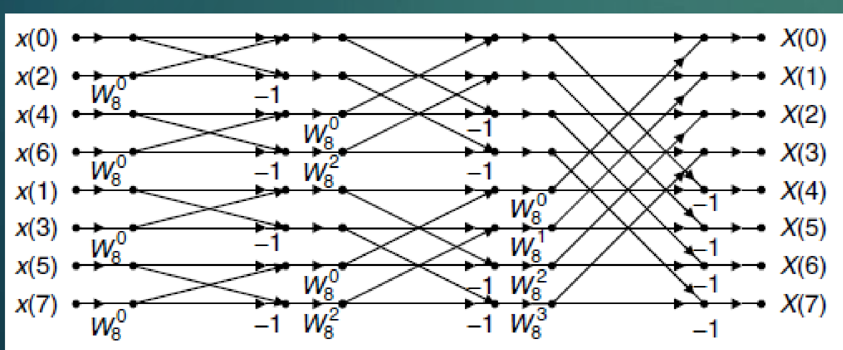
The FFT algorithm is obtained by performing backward iterations. For eight-point FFT:



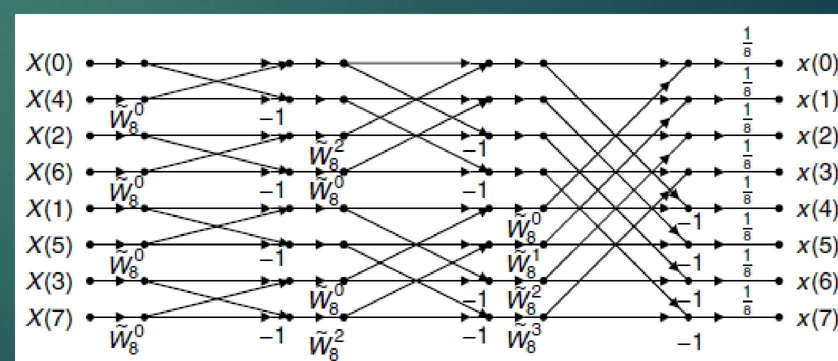
The first backward iteration of eight-point FFT



The first and second backward iteration of eight-point FFT



The eight-point FFT

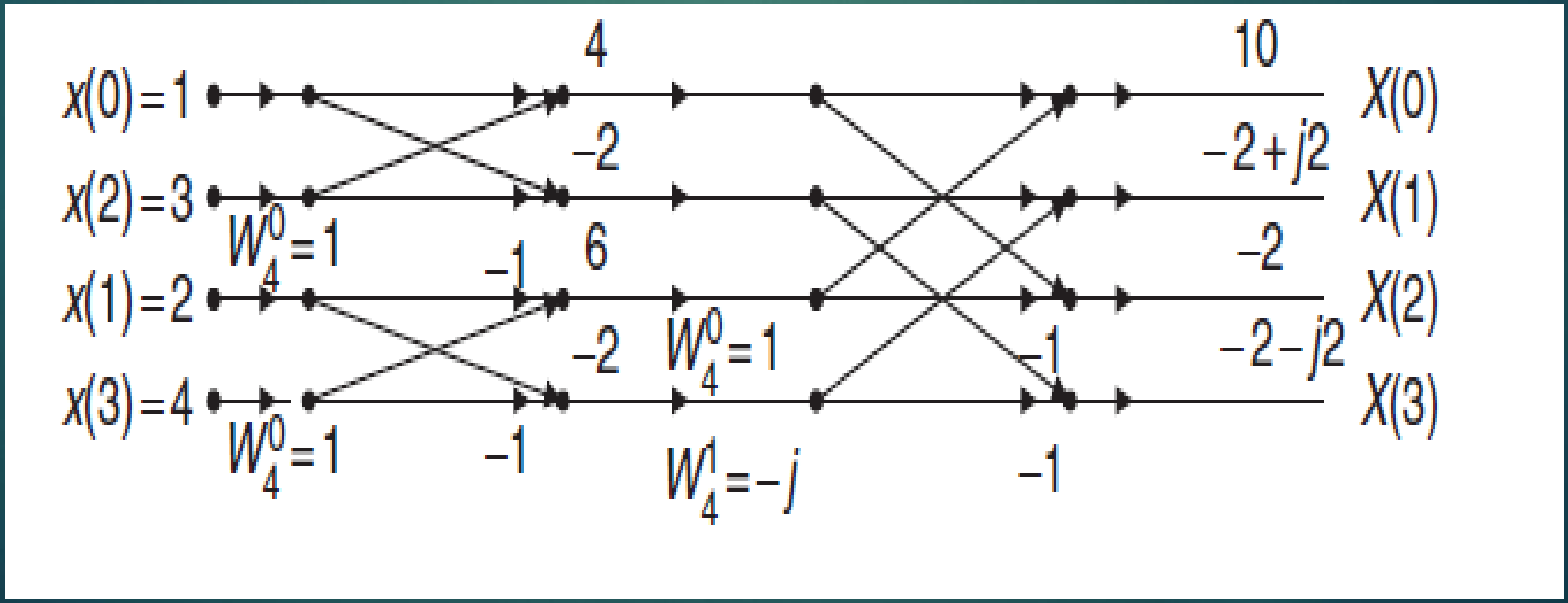


The eight-point inverse FFT

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Method of Decimation-in-Time example(3)

- Given a sequence x(n) for $0 \leq n \leq 3$ where x(0)=1, x(1)=2, x(2)=3 and x(3)=4
- a. Evaluate its DFT x(t) using the decimation in time FFT method
- Solution: block diagram



Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Example Problems

Problem 1.

(i) Solve the following differential equation using Laplace transform:

$$\frac{d^2y(t)}{dt^2} + 12\frac{dy(t)}{dt} + 32y(t) = x(t)$$

$$y(0) = 0 \qquad \frac{dy(0)}{dt} = 0 \qquad x(t) = 32u(t)$$

(ii) Derive the impulse response of the system.

Example Problems

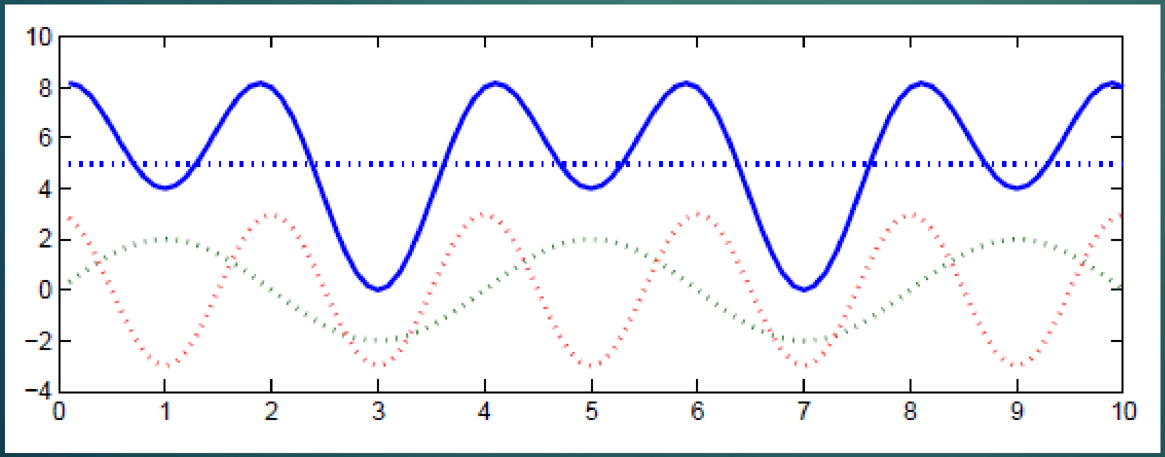
Problem 2.

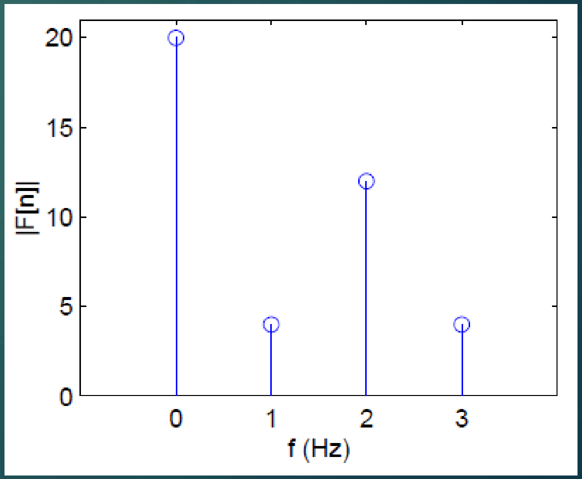
Consider the signal: $x(t) = 5 + 2\cos\left(2\pi t - \frac{\pi}{2}\right) + 3\cos(4\pi t)$

Assuming that $\mathbf{f_s}$=4Hz, we take 4 samples in the first second.

(i) Calculate the DFT coefficients.

(ii) Evaluate the DFT by applying FFT and calculate the speed up.





Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Problems Solutions

Problem 1.

a. $\frac{d^2}{dt^2} + 12\,\frac{d_y}{d_t} + 32y = 32u(t)$

$s^2Y_s + 12sY_{(s)} + 32Y_{(s)} = \frac{32}{s} =>$

$=> Y_{(s)}\,(s^2 + 12s + 32) = \frac{32}{s}$

$\Delta = 144 - 128 = 16 => \sqrt{\Delta} = 4\; r_{1,} = \frac{-12+4}{2} = -4 \qquad r_2 = \frac{-12-4}{2} = -8$

$Y_{(s)} = \frac{32}{s(s+4)(s+8)} = \frac{k_1}{s} + \frac{k_2}{s+4} + \frac{k_3}{s+8}$

$k_1 = \frac{32}{(s+4)\,(s+8)}\Big|_{s=0} = 1 \qquad , k_2 = \frac{32}{s(s+8)}\Big|_{s=-4} = -2 \qquad k_3 = \frac{32}{s(s+4)}\Big|_{s=-8} = 1$

$Y_{(s)} = \frac{1}{s} + \frac{(-2)}{s+4} + \frac{1}{s+8}$

$Y_{(t)} = L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{-2}{s+4}\right) + L^{-1}\left(\frac{1}{s+8}\right) =$

$4(t) - 2e^{-4t}u(t) + e^{-8t}u(t)$

Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Problems Solutions

Problem 1

b. $H(s) = \frac{Y_S}{X_S} = \frac{1}{(s+4)(s+8)}$ $\quad H(s) = \frac{k_1}{s+4} + \frac{k_2}{s+8} = \frac{1}{4}\left(\frac{1}{s+4} - \frac{1}{s+8}\right)$

$k_1 = \frac{1}{(s+8)}\Big|_{s=-4} = \frac{1}{4} \quad k_2 = \frac{1}{(s+4)}\Big|_{s=-8} = -\frac{1}{4}$

$$h(t) = L^{-1}[H_{(s)}] = L^{-1}\left(\frac{\frac{1}{4}}{s+4}\right) + L^{-1}\left(\frac{-\frac{1}{4}}{s+8}\right) \Rightarrow$$

$$h(t) = \frac{1}{4}e^{-4t}u(t) - \frac{1}{4}e^{-8t}u(t)$$

Problems Solutions

Problem 2

$X(t) = 5 + 2\cos\left(2\pi - \frac{\pi}{2}\right) + 3\cos 4\pi t$

$X(n) = 5 + 2\cos\left(\frac{\pi}{2}n - \frac{\pi}{2}\right) + 3\cos \pi n$

$X(k) = \sum_{n=0}^{3} x(n)\, W_N^{kn} = \sum_{n=0}^{3} x(n) e^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{3} x(n) \left(e^{-j\frac{\pi}{2}}\right)^{kn} = \sum_{n=0}^{3} x(n)\,(-j)^{kn}$

$X(0)=x(0)+x(1)+x(2)+x(3)=20$

$X(1)=x(0)+x(1)(-j)+x(2)(-1)+x(3)\, j=-4j$

$X(2)=x(0)+x(1)(-1)+x(2)(1)+x(3)(-1)=-12$

$X(3)=x(0)+x(1)(j)+x(2)(-1)+x(3)(-j)=4j$

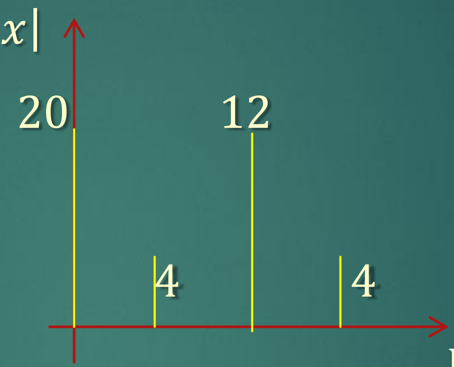
$$\Delta f = \frac{f_s}{N} = 1Hz$$

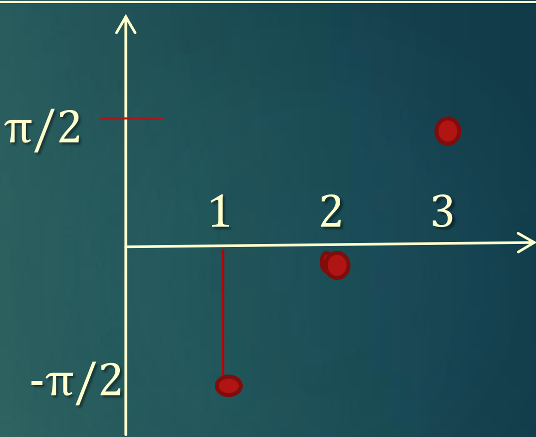
Technological Educational Institute of Crete
Department of Electronic Engineering
Anna Daskalaki | d_anna-13-ntora@windowslive.com

Problems Solutions

Problem 2

x(0) 0 0 0 X(0)
x(1) 1 2 2 X(2)
x(2) 2 1 1 X(1)
x(3)3 3 3 X(3)





X(0)
X(1)
X(2)
X(3)

α(0)=x(0) +x(2)=16 $\alpha'(0)$ $\alpha'(0)+\alpha'(1)$=20

α(1)=x(1)+ x(3) =4 $\alpha'(1)$ $(\alpha'(0)-\alpha'(1))\,W_4^0=12$

b(0) W_4^0 =(x(o) –x(2)) (1) =0 $b'(0)$ $b'(0)+b'(1)$ =-4j

b(1) W_4^1 =(x(0)-x(3)) (-j)=-4j $b'(1)$ $(b'(0)-b'(1))\,W_4^0$= 4j

W_4^0 =1

REFERENCES

- Digital Signal Processing Fundamentals and Applications
  - Li Tan
  - DeVry University
  - Decatur, Georgia
- www.google.com